MATHEMATICS

A TEXTBOOK FOR CLASS VII

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Foreword

THE NATIONAL Policy on Education (NPE) 1986, (revised in 1992) clearly emphasised the need of learning Mathematics as a part of general education. As curriculum renewal is a continuous process, Mathematics curriculum has undergone various changes from time to time in accordance with the changing needs of the technologically oriented society. After a nationwide discussion and consultation, National Council of Educational Research and Training (NCERT) brought out *National Curriculum Framework for School Education* (NCFSE-2000) in November 2000. It reaffirmed the basic principles and directions provided in NPE, 1986 and further elaborated other issues related to Mathematics education at the school stage.

To achieve the general objectives laid down in NCFSE – 2000 which is basically in accordance with NPE 1986, a textbook in Mathematics for Class VI was developed in 2002. The present textbook is a second in that series. In this book also, an attempt has been made to present Mathematics to the pupils through activities and motivating examples relating to their immediate environment. The contents included in the book and the suggested activities have been organised to impart knowledge and develop skills and attitudes on the basis of the changing trends. It is confined mostly to the essentials of Mathematics needed for day-to-day life. Effort has been made to make the textual material and suggested activities suit the diverse needs, background and the environment of the vast school systems in the country. Care has been taken to present the subject matter in a simple and lucid language.

The first draft of the book was developed by a group of experts having a wide experience of teaching and research. The draft was then reviewed in a workshop in which practising teachers, teacher educators and subject experts critically examined the contents and the approach. The comments and suggestions received in the Review Workshop were considered by the authors and the draft was suitably revised and final manuscript developed.

The writing team has made use of suggestions and feedback received from the users of the earlier Mathematics textbook. It has also used references from the earlier textbooks, wherever found suitable in the development of the present book.

I take this opportunity to thank the members of the writing team, its Chairperson, the editors, the reviewers and the institutions they belong to, for preparing this book in such a short time. Suggestions to improve the book would be most welcome.

J.S.RAJPUT
Director
National Council of
Educational Research and Training

New Delhi
February 2003

Preface

MATHEMATICS HAS been an inseparable part of school education since the beginning of formal education and it has played a predominant role not only in the advancement of civilisations but also in the development of physical sciences and other disciplines. As curriculum renewal is a continuous process, Mathematics curriculum has undergone various types of changes from time to time in accordance with the changing needs of the society. The present effort of reforming and updating the curriculum in Mathematics at the Upper Primary Stage is an exercise based on the feedback from users, emergence of new vistas of knowledge and various curricular concerns given in the *National Curriculum Framework for School Education* (NCFSE-2000) brought out by the National Council of Educational Research and Training (NCERT) in November 2000 after in-depth discussions. The draft discussion document on curriculum framework was prepared earlier by various teacher educators, nominees from various examination boards, representatives of the Directorates of Education and the State Councils of Educational Research and Training (SCERTs) of different States/Union Territories, general public and faculty from the universities, colleges, schools and the Council.

Some of the general curricular concerns that have emerged from the *National Curriculum Framework* relevant to teaching of Mathematics at the Upper Primary Stage are as follows :

- Creating an awareness of equality of all with a view to removing prejudices transmitted through social environment and the factor of birth.
- Education for girls.
- Protection of environment.
- Integration of indigenous knowledge and India's contribution to Science and Mathematics from ancient times till date.

- Reduction in the curriculum load by taking out obsolete and redundant contents and providing vertical linkage with Mathematics to be taught at the Secondary Stage.

Keeping in view the above concerns, NCERT constituted writing teams for developing Mathematics textbooks for different stages of schooling. The writing team for the Upper Primary Stage developed a textbook of Mathematics for Class VI last year. The present textbook in Mathematics for Class VII is next in that series This textbook, like the earlier book in the series, takes special care of the following aspects :

- Emergence of new vistas of knowledge.
- Challenges posed for Mathematics by the upcoming edge-cutting technologies.
- Last but not the least, feedback from the users of the earlier curriculum.

A lot of effort has gone into the preparation of the book. First, the drafts prepared by various authors were discussed amongst the team members and modified in the light of the comments and observations made. In these discussions, help of two practising teachers – Ku. Saraswati Kulkarni and Shri Ashok Kumar Gupta was also taken. The modified material was exposed to a group of teachers and experts in a Review Workshop. The manuscript of this ,textbook was finalized keeping in view the comments and suggestions made by the participants of the Review Workshop.

The salient features of the textbook are as follows :

- As far as possible, each topic has been introduced through motivating examples relating to the immediate environment of the pupils.
- The book contains detailed explanations of concepts and a large number of illustrations, solved examples and exercises so that the pupils may grasp the fundamentals as also acquire the desired problem-solving and learning skills.
- A number of learning activities have been suggested for the pupils to rediscover the relevant mathematical facts, and to develop appropriate drawing and measuring skills.

Some of the included word problems develop awareness towards the need for national integration, protection of environment, removal of social barriers, observance of small family norms, elimination of sex-bias and so on. It is expected that the message of these problems would get into the minds of the pupils, and the teachers would be conscious of this while teaching.

- The vocabulary and terminology used in the book is in accordance with the comprehension and maturity level of the pupils.
- A list of important concepts and results has been provided at the end of each Chapter in the form of *'Things to Remember'*.
- Historical references, especially to Indian contributions, have been provided at the end of each Unit under the title *'As History Tells Us'*.

I am thankful to Prof. J. S. Rajput, Director, NCERT, who initiated this challenging activity of tremendous national importance and gave us a chance to return the debt that we owe to our profession towards the improvement of Mathematics Education. I am also thankful to Prof. R. D. Shukla, Head, DESM, for his dynamic leadership, enthusiastic cooperation and generation of facilities as and when required. It goes without saying that my warmest thanks are due to the other members – Prof. B. Deokinandan, Shri Mahendra Shanker, Smt. Srijata Das, Prof. Sunder Lal and Prof. Surja Kumari of the writing team, and the participants of the Review Workshop.

To put a full stop to this somewhat lengthy *Preface*, I must mention the too often repeated warning that no book on any subject can ever be the last word on the subject.

Suggestions and comments for the improvement of this textbook are most welcome. I hope the readers would get as much pleasure in reading the book as we have got in writing the same.

ASHA RANI SINGAL
Chairperson
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CONSTITUTION OF INDIA

Part IV A

Fundamental Duties of Citizens

ARTICLE 51A

Fundamental Duties – It shall be the duty of every citizen of India —

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India,

(d) to defend the country and render national service when called upon to do so;

(e) To promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities, to renounce practices derogatory to the dignity of women,

(f) to value and preserve the rich heritage of our composite culture,

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence,

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement.

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GANDHIJI'S TALISMAN

"I will give you a talisman. Whenever you are in doubt or when the self becomes too much with you, apply the following test :

Recall the face of the poorest and the weakest man whom you may have seen and ask yourself if the step you contemplate is going to be of any use to him. Will he gain anything by it ? Will it restore him to a control over his own life and destiny ? In other words, will it lead to Swaraj for the hungry and spiritually starving millions ?

Then you will find your doubts and your self melting away."

MKGandhi

RATIONAL NUMBERS

CHAPTER 1

1.1 Introduction

In Class VI, we studied natural numbers, whole numbers and integers. In earlier classes, we also studied fractional numbers (or fractions). In this Chapter, we introduce a new number system, viz., the system of rational numbers We shall build up these numbers over fractional numbers. We shall learn to represent these numbers on a number line. The equality of rational numbers, the comparison of two rational numbers and the absolute value of a rational number are other concepts that we shall study in this Chapter. Since the new number system is to be built upon fractional numbers, we begin with recall of fractions.

1.2 Fractions : A Review

Fractions are numbers of the type $\frac{3}{5}, \frac{10}{7}$, etc. which may be written in the form $\frac{p}{q}$, where p and q are natural numbers. For a fractional number $\frac{p}{q}$, p is the *numerator* and q is the *denominator*. A fraction is a *proper fraction* if the numerator is smaller than the denominator. When the numerator is greater, then the fraction is called an *improper fraction*. An improper fraction like $\frac{10}{7}$ can be written as $1\frac{3}{7}$. This number is called a *mixed number* in this form.

If the numbers p and q in a fraction $\frac{p}{q}$ have no common factor other than 1, then $\frac{p}{q}$ is in its *simplest* or *lowest* form. If p and q have some factor, say t, in common, then we may write

$$p = m \times t, \; q = n \times t$$

and $$\frac{p}{q}=\frac{m}{n}$$

In this case, $\frac{p}{q}$ and $\frac{m}{n}$ are said to be *equivalent fractions*. For example,

$$\frac{50}{100}=\frac{1\times 50}{2\times 50}=\frac{1}{2}.$$

Thus, $\frac{1}{2}$ and $\frac{50}{100}$ are equivalent fractions. Likewise, $\frac{300}{900}, \frac{20}{60}, \frac{17}{51}$ are all equivalent fractions.

Every fractional number $\frac{p}{q}$, not in the lowest form, is equivalent to a fractional number $\frac{m}{n}$ in the lowest form. For this, we take the HCF of p and q, and divide p and q by the HCF to get m and n respectively. Fractions $\frac{15}{25}, \frac{51}{85}, \frac{111}{185}, \frac{240}{400}$ are all equivalent to $\frac{3}{5}$ in the lowest form.

If $\frac{p}{q}$ is a fraction, then for any non-zero t, the fraction $\frac{p\times t}{q\times t}$ is equivalent to $\frac{p}{q}$. We use this concept to compare two fractional numbers. If $\frac{p}{q}$ and $\frac{r}{q}$ are two fractions having the same denominator, we compare their numerators and say that $\frac{p}{q}<\frac{r}{q}$ if $p<r$ However, if $\frac{p}{q}$ and $\frac{r}{s}$ are two fractional numbers with different denominators, then we first replace the two fractions by fractions $\frac{p\times s}{q\times s}$ and $\frac{r\times q}{s\times q}$ respectively.

Thus, $\frac{p}{q}=\frac{p\times s}{q\times s}, \frac{r}{s}=\frac{r\times q}{s\times q}$ and these fractions have the same denominator.

Now, we compare the numerators $p \times s$ and $r \times q$, and say that

(i) $\frac{p}{q} < \frac{r}{s}$, if $p \times s < r \times q$,

(ii) $\frac{p}{q} = \frac{r}{s}$, if $p \times s = r \times q$, and

(iii) $\frac{p}{q} > \frac{r}{s}$, if $p \times s > r \times q$.

Thus, for example, $\frac{4}{11} < \frac{6}{11}$, $\frac{5}{9} = \frac{35}{63}$ and $\frac{3}{5} > \frac{4}{7}$.

The fractions can be represented on the number line. Suppose, we wish to represent $\frac{3}{8}$ on the number line. We draw a line and mark on it the points O corresponding to the whole number zero and P corresponding to the number 3. We now divide the line segment OP into 8 equal parts. For this, we bisect OP at D. Then we bisect OD at B and finally we bisect OB at A.

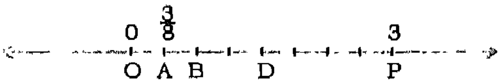


Fig. 1.1

Here, the length of the line segment OA is $\frac{1}{8}$ of the length of the line segment OP. Since the line segment OP represents 3, the line segment OA would represent $\frac{1}{8}$ of 3, i.e., $\frac{3}{8}$. Thus, with O representing the number zero, the point A will represent the fractional number $\frac{3}{8}$.

In general, every fractional number $\frac{p}{q}$ can be represented on the number line,

1.3 Need for Rational Numbers

In Class VI, you have learnt to solve some simple equations in one variable. Suppose we have an equation $2x - 4 = 0$.

You can verify that the integer 2 is a solution of this equation. However, if we have an equation $2x - 3 = 0$, then no integer value of x can satisfy the equation. It is only a fraction $x = \frac{3}{2}$ which is a solution of this equation.

Now we consider the equation $2x + 3 = 0$.

Do we have an integral or a fractional value of x which satisfies this equation? No. This equation has no solution in integers or in fractions. We need to extend our number systems (of integers and fractions) to include numbers in which equations like this may have solutions.

We may look at the need for extending integers and fractions from another view point. We have number -3 on the number line denoted by, say P′. As we divide the line segment OP into eight parts, we may divide the segment OP′ also into eight parts, with points, say A′, B′, ... What does A′ represent ? As A represents $\frac{3}{8}$, A′ represents $\frac{-3}{8}$. However, this number $\frac{-3}{8}$ is neither an integer nor a fraction. So, we need to extend our number system to include the 'fraction like numbers' of the type $\frac{-3}{8}$.

1.4 Rational Numbers

Let us extend the system of fractions and consider numbers of the form $\frac{p}{q}$, where p and q are any integers, but q is always non-zero. Thus, we consider all integers, all fractions and all 'fraction like numbers' such as

$$\frac{-4}{3}, \frac{3}{-4}, \frac{13}{-9}, \frac{16}{2}, \frac{-28}{-51}, \frac{0}{100}$$

etc. However, we do not include 'numbers' like $\frac{5}{0}, \frac{97}{0}$, where zero is in the denominator. Such extended numbers are called the system of *rational numbers*. Thus, *a rational number is a number of the form* $\frac{p}{q}$, *where p and q are integers and* $q \neq 0$.

The integer p in the rational number $\frac{p}{q}$ is called its *numerator* and q is called its *denominator*. $\frac{-3}{4}$ is a rational number with numerator –3 and denominator 4, whereas $\frac{3}{-4}$ is a rational number with numerator 3 and denominator – 4. Likewise, 0 is the numerator of the rational number $\frac{0}{100}$ and 1 is the denominator of the rational number $\frac{16}{1}$.

It follows from the definition itself that every fraction is a rational number. However, not every rational number is a fraction (e.g., $\frac{-3}{4}$). In fact, a rational number with numerator and denominator both positive, is a fraction.

Just as a positive integer x is regarded as a fraction $\frac{x}{1}$ with x as numerator and 1 as denominator, so any integer y (positive, negative or zero) may be regarded as a rational number $\frac{y}{1}$ with numerator y and denominator 1. Thus, the integers 100, –6, 0 are rational numbers, because these can be represented as $\frac{100}{1}, \frac{-6}{1}$ and $\frac{0}{1}$ respectively.

Two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ are said to be *equivalent*, if

$$p \times s = r \times q$$

Thus, rational numbers $\frac{1}{2}$ and $\frac{2}{4}$ are equivalent rational numbers. As numbers, equivalent rational numbers represent the same number.

Thus, numbers $\frac{15}{5}$ and $\frac{90}{30}$ represent the same rational number $\frac{3}{1}$,

Recall that if $\frac{p}{q}$ is a fraction, then the fraction $\frac{2p}{2q}$ is equivalent to $\frac{p}{q}$.

Similarly, if $\frac{p}{q}$ is a rational number, then $\frac{2p}{2q}$ is also a rational number equivalent to $\frac{p}{q}$. Likewise $\frac{100p}{100q}$, $\frac{m \times p}{m \times q}$, $m \neq 0$, are all rational numbers equivalent to $\frac{p}{q}$.

$\frac{-30}{16}$ is a rational number equivalent to $\frac{-15}{8}$. All rational numbers $\frac{42}{24}, \frac{21}{12}, \frac{14}{8}, \frac{7}{4}, \frac{-14}{-8}, \frac{-21}{-12}, \frac{-42}{-24}$ are equivalent to $\frac{7}{4}$.

Like a fraction, a rational number $\frac{p}{q}$ is said to be in the *simplest* or *lowest form*, if p and q have no common factor other than 1, or that the HCF of p and q is 1. The rational numbers

$$\frac{3}{7}, \frac{-103}{5}, \frac{19}{-115}$$

are all rational numbers in the lowest form. However, $\frac{15}{70}$ is not in the lowest form, because 5 is a factor common to both numerator and denominator. To find the corresponding lowest form, we write $\frac{15}{70} = \frac{3 \times 5}{2 \times 5 \times 7} = \frac{3}{2 \times 7} = \frac{3}{14}$ and thus $\frac{3}{14}$ is in the lowest form. *Every rational number* $\frac{p}{q}$ *can be put in the lowest form.* For this, we take the following steps:

***Step* 1:** Find the HCF m of p and q. If $m = 1$, then $\frac{p}{q}$ is in the lowest form.

***Step* 2:** If $m \neq 1$, then find $p' = p \div m$ and $q' = q \div m$. The rational number $\frac{p'}{q'}$ is the lowest form of $\frac{p}{q}$.

Example 1 : Find whether the following numbers are in the lowest form. Find the lowest form of those of the numbers which are not in the lowest form.

(i) $\frac{12}{16}$ (ii) $\frac{17}{79}$ (iii) $\frac{-24}{36}$ (iv) $\frac{-60}{-72}$

Solution : (i) Here, $12 = 2 \times 2 \times 3$, and

$16 = 2 \times 2 \times 2 \times 2$

Thus, HCF of 12 and $16 = 2 \times 2 = 4$ (Step 1)

Therefore, $\frac{12}{16}$ is not in the lowest form.

Also, $12 \div 4 = 3$, $16 \div 4 = 4$ (Step 2)

Therefore, $\frac{12}{16} = \frac{3}{4}$ is in the lowest form.

(ii) Here, HCF of 17 and 79 = 1

Therefore, $\frac{17}{79}$ is in the lowest form.

(iii) The HCF of 24 and 36 = 12

Thus, $\frac{-24}{36}$ is not in the lowest form. However,

$$\frac{-24}{36}=\frac{-2\times12}{3\times12}=\frac{-2}{3},$$

which is in the lowest form.

(iv) Here, HCF of 60 and 72 = 12, and

$$\frac{-60}{-72}=\frac{-5\times12}{-6\times12}=\frac{-5}{-6}=\frac{5}{6}$$

As the numerator and denominator of a rational number can be divided by a common factor, if any, to get an equivalent rational number, so the numerator and denominator can be multiplied by the same number to get an equivalent rational number. Thus,

$$\frac{2}{3},\frac{4}{6}\left(=\frac{2\times2}{3\times2}\right),\frac{-8}{-12}\left(=\frac{2\times(-4)}{3\times(-4)}\right) \text{ etc.}$$

are all equivalent and represent the same rational number. One consequence of this property is the following :

Any rational number $\frac{p}{q}$ *may be replaced by an equivalent rational number* $\frac{p\times k}{q\times k}$, *where k is any positive or negative integer.*

Example 2 : Fill in the blanks :

(i) $\frac{5}{-7}=\frac{....}{35}=\frac{....}{-77}$

(ii) $\frac{7}{13}=\frac{35}{....}=\frac{-63}{....}$

(iii) $\frac{90}{165}=\frac{-6}{....}=\frac{....}{-55}$

Solution : (i) 35 ÷ (–7) = –5 , i.e., 35 = (–7) ×(–5)

$$\therefore \quad \frac{5}{-7}=\frac{5\times(-5)}{(-7)\times(-5)}=\frac{-25}{35}$$

Similarly, $(-77) \div (-7) = 11$, i.e., $(-7) \times 11 = -77$

$$\therefore \quad \frac{5}{-7} = \frac{5 \times 11}{(-7) \times 11} = \frac{55}{-77}$$

$$\text{Thus,} \quad \frac{5}{-7} = \frac{-25}{35} = \frac{55}{-77}$$

(ii) Here, $35 = 7 \times 5$

$$\therefore \quad \frac{7}{13} = \frac{7 \times 5}{13 \times 5} = \frac{35}{65}$$

Also, $-63 = 7 \times (-9)$

$$\therefore \quad \frac{7}{13} = \frac{7 \times (-9)}{13 \times (-9)} = \frac{-63}{-117}$$

Thus, $\frac{7}{13} = \frac{35}{65} = \frac{-63}{-117}$

(iii) Note that in (i) and (ii), the numbers $\frac{5}{-7}$ and $\frac{7}{13}$ were already in the lowest form.

Here, we first put $\frac{90}{165}$ in the lowest form.

Now $90 = 2 \times 3 \times 3 \times 5$, and

$165 = 3 \times 5 \times 11$

Therefore, HCF of 90 and 165 = 15

Also, $\frac{90}{165} = \frac{6 \times 15}{11 \times 15} = \frac{6}{11}$

Thus, the rational number $\frac{90}{165}$ in the lowest form equals $\frac{6}{11}$.

Also, $\frac{6}{11}=\frac{-6}{-11}$

Further, $-55 \div 11 = -5$

Therefore, $\frac{6}{11}=\frac{6\times(-5)}{11\times(-5)}=\frac{-30}{-55}$

Thus, $\frac{90}{165}=\frac{-6}{-11}=\frac{-30}{-55}$

Remarks : 1. A rational number in the lowest form is said to be in the *standard form*, if the denominator is positive. Thus, the standard form of $\frac{-26}{39}$ is $\frac{-2}{3}$.

The number $\frac{3}{-4}$, though in the lowest form, is not in the standard form. We convert it into standard form as $\frac{3}{-4}=\frac{3\times(-1)}{-4\times(-1)}=\frac{-3}{4}$.

2. We observe above that $\frac{3}{-4}$ and $\frac{-3}{4}$ are equivalent rational numbers. In fact, if p and q are any positive integers, then $\frac{p}{-q}=\frac{p\times(-1)}{-q\times(-1)}=\frac{-p}{q}$ and $\frac{-p}{-q}=\frac{-p\times(-1)}{-q\times(-1)}=\frac{p}{q}$, i.e., the rational number $\frac{p}{-q}$ is equivalent to $\frac{-p}{q}$ and $\frac{-p}{-q}$ is equivalent to $\frac{p}{q}$. In other words, any rational number with negative denominator may be replaced by an equivalent rational number with positive denominator. In view of this fact, we may also say that *a rational number is a number of the form $\frac{p}{q}$, where p is any integer, but q is a positive integer.*

EXERCISE 1.1

Write the numerator of each of the following rational numbers :

(i) $\frac{12}{23}$ (ii) $\frac{-27}{53}$ (iii) $\frac{99}{-1000}$ (iv) $\frac{1}{101}$ (v) $\frac{-67}{-167}$

Write the denominator of the rational numbers in the question above.

In each of the following cases, write the rational number whose numerator and denominator are respectively given below :

(i) 2^3 and 3^2 (ii) $5 - 49$ and $55 - 9$

(iii) $28 + 79$ and $79 - 28$ (iv) 5×3 and $16 \div 8$

Write the following rational numbers in the lowest form :

(i) $\frac{2}{10}$ (ii) $\frac{-36}{180}$ (iii) $\frac{-64}{256}$

(iv) $\frac{91}{364}$ (v) $\frac{24}{64}$ (vi) $\frac{44}{428}$

Express $\frac{1}{4}$ as a rational number with denominator

(i) 20 (ii) 36 (iii) –80 (iv) –100 (v) 40000

Write $\frac{2}{5}$ in an equivalent form so that the numerator may be equal to

(i) –56 (ii) 154 (iii) –750 (iv) 500 (v) –6250

Fill in the blanks :

(i) $\frac{2}{3} = \frac{\ldots}{135}$ (ii) $\frac{\ldots}{4} = \frac{90}{120}$ (iii) $\frac{5}{\ldots} = \frac{90}{216}$ (iv) $\frac{9}{16} = \frac{90}{\ldots}$

In each of the following, find an equivalent form of the rational numbers having a common denominator :

(i) $\frac{5}{6}$ and $\frac{7}{9}$ (ii) $\frac{2}{3}, \frac{5}{6}$ and $\frac{7}{12}$

(iii) $\frac{4}{5}, \frac{17}{20}, \frac{23}{40}$ and $\frac{11}{16}$ (iv) $\frac{5}{7}, \frac{3}{8}, \frac{9}{14}$ and $\frac{20}{21}$

9. Express in the standard form :

(i) $\frac{-144}{-504}$ (ii) $\frac{140}{490}$

(iii) $\frac{-132}{-330}$ (iv) $\frac{240}{-840}$

10. Write true (T) or false (F) for the following statements .

(i) $\frac{-3}{5}$ is a fraction

(ii) $2\frac{7}{9}$ is a mixed number.

(iii) $\frac{24}{44}$ is equivalent to $\frac{2}{4}$.

(iv) $\frac{150}{1500}$ is equivalent to $\frac{1}{10}$.

(v) $\frac{-2}{3} < \frac{3}{-4}$, since $-2 \times (-4) < 3 \times 3$.

(vi) Equation $8x + 8 = 0$ can be solved in integers.

(vii) Equation $8x + 4 = 0$ can be solved in fractions.

(viii) $\frac{1}{0}$ is not a rational number.

(ix) Rational number $\frac{1}{9}$ is in the lowest form, but $\frac{9}{1}$ is not in the lowest form.

(x) If a rational number $\frac{p}{q}$ is in the standard form, then $\frac{q}{p}$ is also in the standard form.

1.5 Rational Numbers on the Number Line

Let us recall from Class VI, the method of bisecting a line segment AB

Let AB be the given line segment. With A as centre and radius greater than half the length AB, draw two arcs on both sides of AB With the same radius, but centre at B, draw two more arcs on the two sides of AB intersecting the earlier two arcs at P and Q. Join the points P and Q cutting the line

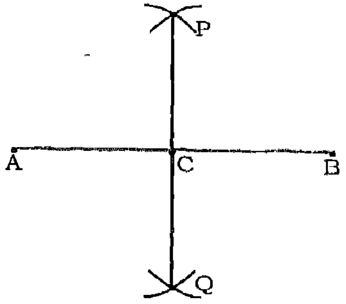


Fig. 1.2

segment AB at C (Fig. 1.2). The point C is the mid-point of AB, i.e., the line segment AB is bisected at C.

Repeating the method of bisection, we can divide AB into $2 \times 2 = 4$ equal parts, $2 \times 4 = 8$ equal parts, $2 \times 8 = 16$ equal parts, etc. Can we divide AB into 64 equal parts or into 256 equal parts?

Can we divide a line segment into 3 equal parts? We can. In fact, we may divide a line segment into any number of equal parts. We do not go into the details of the construction, but we make use of this fact to represent a rational number on the number line.

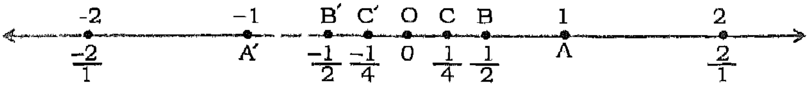


Fig. 1.3

Suppose, we wish to represent rational numbers like $\frac{2}{1}, \frac{-2}{1}$ on the number line. We have already observed that $\frac{2}{1}$ is the same as integer 2 and $\frac{-2}{1}$ is the same as integer -2. Since we have already represented integers on the number line, therefore all rational numbers with denominator 1 have a representation on the number line.

Suppose now that we wish to represent $\frac{1}{2}$ or $\frac{1}{4}$ on the number line (Fig. 1.3). For this, we take the point A on the line which represents 1 and consider the line segment OA (O being the point which corresponds to zero on the line). We bisect the line segment OA. If B is the point of bisection, then the length of the line segment OB is $\frac{1}{2}$ and, therefore, B represents $\frac{1}{2}$. By bisecting the line segment OB, we get a point C which represents the rational number $\frac{1}{4}$. Similarly, if A′ represents the number -1, then the point B′ which bisects the line segment OA′ represents $\frac{-1}{2}$ and the rational number $\frac{-1}{4}$ is represented by

C′, the point of bisection of the line segment OB′. By repeated bisection, we may represent on the number line any rational number with numerator 1 and the denominator a power of 2.

Remarks : 1. Though we discussed above the method of representing a special class of rational numbers, viz., the rationals of the type $\frac{1}{4}, \frac{-1}{4}, \frac{1}{2}, \frac{-1}{2}$ etc., but the fact is that *any rational number may be represented on the number line.* This is possible, because a given line segment may be divided into *any* number of equal parts. Thus, if we wish to represent $\frac{12}{7}$ on the number line, we choose the point, say P, corresponding to the integer 12 on the line (Fig. 1.4). We divide the line segment OP into seven equal parts and mark the points of division on the line. If Q is the first point on the right of O, then the length of the line segment OQ is $\frac{1}{7}$th part of the length of the line segment OP Since length OP denotes 12 units, therefore OQ denotes $\frac{1}{7}$th of 12, i.e., $\frac{12}{7}$ units. In other words, Q represents the number $\frac{12}{7}$.

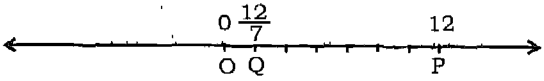


Fig 1.4

Suppose now that we wish to represent on this line, rational number of the type $\frac{-17}{9}$.

This time we find the point P on the left of O which represents the integer – 17. Now, divide the line segment OP into nine equal parts. If Q is the first corresponding point on the left of O, then the point Q represents the number $\frac{-17}{9}$.

2. To represent a rational number on the number line, it is better to first express it in the standard form because to represent $\frac{p}{q}$ on the line, we shall have to divide a line segment of length p into q equal parts. Thus, q has to be positive. Further, consider the rational numbers $\frac{1}{2}$ and $\frac{50}{100}$. If we represent $\frac{1}{2}$ and $\frac{50}{100}$ on the number line, we get the same point. However, to represent $\frac{1}{2}$, we need two equal divisions, whereas in $\frac{50}{100}$, we have to divide 50 units long line segment into 100 equal divisions.

We know that 0 is a rational number, because it can be written as $\frac{p}{q}$, where $p = 0$ and q is any non-zero integer. Thus, $0 = \frac{0}{1} = \frac{0}{-100} = \frac{0}{596}$, etc. The numbers which lie on the right hand side of 0 are all positive numbers. Thus, $\frac{1}{9}$, $\frac{-15}{-19}, \frac{16}{87}, \frac{-106}{-141}$ are all positive numbers. Numbers which lie on the left side of 0 are all negative numbers. Therefore, $\frac{-3}{7}, \frac{-8}{139}, \frac{6}{-593}, \frac{802}{-9999}$ are all negative numbers. Thus, for a rational number $\frac{p}{q}$, we see that

(i) $\frac{p}{q}$ is zero, if p is zero.

(ii) $\frac{p}{q}$ is positive, if p and q are both positive or p and q are both negative.

(iii) $\frac{p}{q}$ is negative, if either p is negative and q is positive or p is positive and q is negative,i.e., if p and q have opposite signs.

1.6 Order Relation

Consider two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$. We represent them on the number line. If they correspond to the same point on the line, then

$$\frac{p}{q}=\frac{r}{s}$$

However, if $\frac{p}{q}\neq\frac{r}{s}$, then $\frac{p}{q}$ lies either to the left of $\frac{r}{s}$ or to the right of $\frac{r}{s}$.

If $\frac{p}{q}$ lies to the left of $\frac{r}{s}$, then

$$\frac{p}{q}<\frac{r}{s}$$

If $\frac{p}{q}$ lies to the right of $\frac{r}{s}$, then

$$\frac{p}{q}>\frac{r}{s}$$

Thus, for two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, we have either

(i) $\frac{p}{q}=\frac{r}{s}$ or (ii) $\frac{p}{q}<\frac{r}{s}$ or (iii) $\frac{p}{q}>\frac{r}{s}$

Without representing the rational numbers on the number line, we compare them by the method similar to the one used for fractional numbers.

If $\frac{p}{q}$ and $\frac{r}{s}$ (both in standard form) have equal denominator, i.e., if $s = q$, then we compare their numerators.

If $p = r$, then $\frac{p}{q}=\frac{r}{s}$;

if $p < r$, then $\frac{p}{q}<\frac{r}{s}$; and

if $p > r$, then $\frac{p}{q}>\frac{r}{s}$.

Thus, to compare $\frac{3}{25}$ and $\frac{7}{25}$, we compare 3 and 7.

Since $3 < 7$, therefore $\frac{3}{25} < \frac{7}{25}$.

Similarly, to compare $\frac{-7}{19}$ and $\frac{-11}{19}$, we compare -7 and -11

Since $-7 > -11$, therefore $\frac{-7}{19} > \frac{-11}{19}$.

Thus, for two rational numbers $\frac{p}{q}$ and $\frac{r}{q}$ with the same positive denominator, we have

$$\boxed{\frac{p}{q} = \frac{r}{q},\ \frac{p}{q} < \frac{r}{q} \text{ or } \frac{p}{q} > \frac{r}{q} \text{ according as } p = r,\ p < r \text{ or } p > r}$$

How do we compare rational numbers with unequal denominators? If $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers and $q \neq s$, then we replace $\frac{p}{q}$ and $\frac{r}{s}$ by equivalent rational numbers with the *same* (positive) denominators and then compare them.

For example, to compare $\frac{3}{5}$ and $\frac{7}{9}$, we replace

$\frac{3}{5}$ by $\frac{3\times 9}{5\times 9} = \frac{27}{45}$, and

$\frac{7}{9}$ by $\frac{7\times 5}{9\times 5} = \frac{35}{45}$

Now we compare $\frac{27}{45}$ and $\frac{35}{45}$. Since $27 < 35$, therefore we have $\frac{27}{45} < \frac{35}{45}$ as earlier, i.e., $\frac{3}{5} < \frac{7}{9}$.

Example 3 : Compare the rational numbers

(i) $\frac{21}{15}$ and $\frac{19}{27}$ (ii) $\frac{-15}{22}$ and $\frac{-34}{29}$

(iii) $\frac{111}{-5}$ and $\frac{150}{-7}$ (iv) $\frac{13}{-8}$ and $\frac{-27}{12}$

Solution : (i) $\frac{21}{15} = \frac{21\times 27}{15\times 27} = \frac{567}{405}$, and

$$\frac{19}{27} = \frac{19\times 15}{27\times 15} = \frac{285}{405}$$

Since 567 > 285, therefore $\frac{21}{15} > \frac{19}{27}$.

(ii) $\frac{-15}{22} = \frac{-15\times 29}{22\times 29} = \frac{-435}{638}$ and

$$\frac{-34}{29} = \frac{-34\times 22}{29\times 22} = \frac{-748}{638}$$

Since –748 < – 435 , therefore $\frac{-34}{29} < \frac{-15}{22}$.

(iii) $\frac{111}{-5} = \frac{111\times(-7)}{-5\times(-7)} = \frac{-777}{35}$ and

$$\frac{150}{-7} = \frac{150\times(-5)}{-7\times(-5)} = \frac{-750}{35}$$

Since – 777 < – 750, therefore $\frac{111}{-5} < \frac{150}{-7}$.

(iv) $\frac{13}{-8} = \frac{13\times 12}{-8\times 12} = \frac{156\times(-1)}{-96\times(-1)} = \frac{-156}{96}$ and

$$\frac{-27}{12} = \frac{-27\times(-8)}{12\times(-8)} = \frac{216\times(-1)}{-96\times(-1)} = \frac{-216}{96}$$

Since – 156 > –216, therefore $\frac{13}{-8} > \frac{-27}{12}$.

In all the above examples, we converted numbers into equivalent rational numbers having *common positive denominator.* For this, we took the product of the two denominators (and – 1 when the product was negative). In example 3 (i), the common denominator was 15 × 27 while in example 3 (iv), the common denominator was 96 [= –8 × 12 × (–1)]. It is not advisable to take such large numbers as common denominator. We may take the common denominator as the least common multiple (LCM) of the denominators. For example, in (i), instead of taking the common denominator as 15 × 27 = 405, we could have taken it as 135.

Then, $\frac{21}{15} = \frac{21 \times 9}{15 \times 9} = \frac{189}{135}$ and

$$\frac{19}{27} = \frac{19 \times 5}{27 \times 5} = \frac{95}{135}$$

Since 189 > 95, therefore $\frac{21}{15} > \frac{19}{27}$.

Example 4 : Fill in the blanks by < or > :

(i) $\frac{5}{8} \ldots \frac{11}{12}$ (ii) $\frac{-7}{60} \ldots \frac{5}{-40}$ (iii) $\frac{-4}{9} \ldots \frac{-3}{-7}$

Solution : (i) The LCM of 8 and 12 = 24. Also, 24 = 8 × 3 = 12 × 2

$$\therefore \quad \frac{5}{8} = \frac{5 \times 3}{8 \times 3} = \frac{15}{24}, \text{ and}$$

$$\frac{11}{12} = \frac{11 \times 2}{12 \times 2} = \frac{22}{24}.$$

Since 15 < 22, therefore $\frac{5}{8} < \frac{11}{12}$.

(ii) Here, LCM of 60 and 40 = 120. Also, $120 = 60 \times 2 = -40 \times (-3)$

$$\therefore \quad \frac{-7}{60} = \frac{-7\times2}{60\times2} = \frac{-14}{120}, \text{ and}$$

$$\frac{5}{-40} = \frac{5\times(-3)}{-40\times(-3)} = \frac{-15}{120}$$

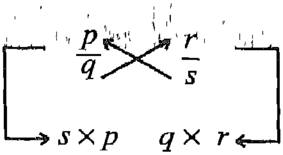
Since $-14 > -15$, therefore $\frac{-7}{60} > \frac{5}{-40}$.

(iii) Since $\frac{-4}{9}$ is a negative rational number, and $\frac{-3}{-7} = \frac{3}{7}$ is a positive rational number, therefore $\frac{-4}{9} < \frac{-3}{-7}$.

There is yet another procedure to compare two rational numbers with unequal denominators To compare two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$, we proceed as follows :

(i) Write the two numbers $\frac{p}{q}$ and $\frac{r}{s}$ in the standard form in a line.

(ii) Below $\frac{p}{q}$, write the integer $s \times p$ and below the number $\frac{r}{s}$, write the integer $q \times r$.



(iii) Compare the integers $s \times p$ and $q \times r$. If $s \times p < q \times r$, then $\frac{p}{q} < \frac{r}{s}$.

Example 5 : Compare the following pairs of rational numbers :

(i) $\frac{3}{7}$ and $\frac{5}{11}$ (ii) $\frac{-3}{8}$ and $\frac{-7}{19}$ (iii) $\frac{-17}{23}$ and $\frac{5}{-8}$

Solution : (i)

$$\begin{array}{cc} \dfrac{3}{7} & \dfrac{5}{11} \\ 11 \times 3 & 7 \times 5 \\ (33) & (35) \end{array}$$

Since 33 < 35, therefore $\frac{3}{7} < \frac{5}{11}$.

(ii)

$$\begin{array}{cc} \dfrac{-3}{8} & \dfrac{-7}{19} \\ 19 \times (-3) & 8 \times (-7) \\ (-57) & (-56) \end{array}$$

Since $-57 < -56$, therefore $\frac{-3}{8} < \frac{-7}{19}$.

(iii) First we put $\frac{5}{-8}$ in the standard form as $\frac{-5}{8}$.

Now

$$\begin{array}{cc} \dfrac{-17}{23} & \dfrac{-5}{8} \\ 8 \times (-17) & 23 \times (-5) \\ (-136) & (-115) \end{array}$$

Since $-136 < -115$, therefore $\frac{-17}{23} < \frac{-5}{8}$.

1.7 Absolute Value of a Rational Number

Recall that the absolute value of an integer is an integer. If the integer is positive or zero, the absolute value is the integer itself, whereas the absolute value of a negative integer is the integer without the negative sign. Thus, the absolute value of the integer 5, i.e., | 5 | is 5, whereas the absolute value | − 9 | of the integer − 9 is the integer 9. Likewise, | −101 | = 101, | 99 | = 99, | 0 | = 0, etc.

What about the absolute value of a rational number ? As is the case with integers, the absolute value of a rational number is also a rational number. The numerator of the absolute value of a rational number is the absolute value of the numerator of the rational number. Likewise the denominator of the absolute value is the absolute value of the denominator. In other words, if the rational number x is of the form $\frac{p}{q}$, then the absolute value of x, i.e., $|x|$ is $\frac{|p|}{|q|}$. Thus,

$$\left|\frac{3}{5}\right| = \frac{|3|}{|5|} = \frac{3}{5},$$

$$\left|\frac{-3}{5}\right| = \frac{|-3|}{|5|} = \frac{3}{5},$$

$$\left|\frac{61}{-9}\right| = \frac{|61|}{|-9|} = \frac{61}{9},$$

$$\left|\frac{-73}{-87}\right| = \frac{|-73|}{|-87|} = \frac{73}{87},$$

$$\left|\frac{-173}{209}\right| = \frac{|-173|}{|209|} = \frac{173}{209}.$$

EXERCISE 1.2

1. Draw the number line and represent the following rational numbers on it :

(i) $\frac{3}{4}$ (ii) $\frac{3}{8}$ (iii) $\frac{5}{8}$ (iv) $\frac{3}{16}$

2. Let O, P and Z represent the numbers 0, 3 and –5 respectively on the number line (Fig 1.5). Mark three points Q, R and S between O and P such that OQ = QR = RS = SP. What are the rational numbers represented by the points Q, R and S? Now choose a point T between Z and O so that ZT = TO. Which rational number does T represent?

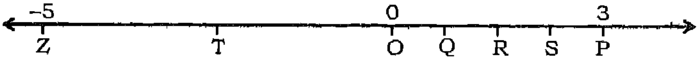


Fig. 1.5

3. Which of the following pairs of rational numbers are equal?

(i) $\frac{-9}{12}$ and $\frac{8}{-12}$ (ii) $\frac{-16}{20}$ and $\frac{20}{-25}$

(iii) $\frac{-7}{21}$ and $\frac{3}{9}$ (iv) $\frac{-8}{-14}$ and $\frac{13}{21}$

4. Which of the two rational numbers is greater?

(i) $\frac{-4}{11}, \frac{3}{11}$ (ii) $\frac{-5}{8}, \frac{-3}{4}$

(iii) $\frac{-7}{12}, \frac{5}{-8}$ (iv) $\frac{-4}{9}, \frac{-3}{-7}$

5. Which of the two rational numbers is smaller?

(i) $\frac{-4}{7}, \frac{5}{-7}$ (ii) $\frac{6}{13}, \frac{-7}{-13}$

(iii) $\frac{16}{-5}, 3$ (iv) $\frac{4}{-3}, \frac{-8}{7}$

6. Fill in the blanks by the correct symbol out of >, = and < :

(i) $\frac{-5}{7} \dots \frac{6}{13}$ (ii) $\frac{-4}{5} \dots \frac{-5}{6}$

(iii) $\frac{-7}{8} \dots \frac{21}{-24}$ (iv) $\frac{-9}{-10} \dots \frac{8}{9}$

7. Find the absolute values of the rational numbers given in Q.4 above.

8. Find the absolute value of each of the rational numbers given in Q.5 above and verify that the absolute value of each number is greater than or equal to the number.

9. Fill in the blanks by the correct symbol >, = or < :

(i) $\left|\frac{-5}{7}\right| \dots \left|\frac{6}{13}\right|$ (ii) $\left|\frac{-4}{5}\right| \dots \left|\frac{-5}{6}\right|$

(iii) $\left|\frac{-7}{8}\right| \dots \left|\frac{21}{-24}\right|$ (iv) $\left|\frac{-9}{-10}\right| \dots \left|\frac{8}{9}\right|$

Compare the results with the results in Q.6 above.

10. Write true (T) or false (F) for the following statements :

(i) The rational number $\frac{57}{23}$ lies to the left of zero on the number line.

(ii) The rational number $\frac{7}{-4}$ lies to the right of zero on the number line.

(iii) The rational number $\frac{-8}{-3}$ lies neither to the right nor to the left of zero on the number line

(iv) The rational numbers $\frac{1}{2}$ and -1 are on opposite sides of zero on the number line.

(v) If $\frac{p}{q} < \frac{r}{s}$, then $\left|\frac{p}{q}\right| < \left|\frac{r}{s}\right|$.

(vi) If $|x| = |y|$, then $x = y$.

(vii) If $|x| = 0$, then $x = 0$.

Things to Remember

1. A number of the form $\frac{p}{q}$, where p and q are positive integers, is called a fractional number.

2. A number of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is called a rational number.

3. A rational number $\frac{p}{q}$ is said to be in the lowest form, if HCF of p and q is 1.

4. A rational number $\frac{p}{q}$ in the lowest form is said to be in the standard form if $q > 0$.

5. Every integer is a rational number and every fraction is a rational number .

6. If $\frac{p}{q}$ is a rational number and m is a non-zero integer, then $\frac{p}{q} = \frac{p \times m}{q \times m}$.

7. If $\frac{p}{q}$ is a rational number and m is a common divisor of p and q, then $\frac{p}{q} = \frac{p \div m}{q \div m}$.

8. If $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers with q and s positive, then $\frac{p}{q} > \frac{r}{s}$

if $s \times p > q \times r$

9. If x and y are two rational numbers, then

either (i) $x > y$ or (ii) $x = y$ or (iii) $x < y$.

10. Every rational number can be represented on a number line.

11. Every rational number has an absolute value which is greater than or equal to zero.

12. The absolute value of a rational number is greater than or equal to the number itself.

OPERATIONS ON RATIONAL NUMBERS

CHAPTER 2

2.1 Introduction

In Class VI, we studied various operations on numbers like addition of natural numbers, subtraction of whole numbers, multiplication and division of integers, etc. We also studied properties of these operations. In this Chapter, we shall extend the scope of these operations to rational numbers. We shall then derive some properties of these operations. Most of these properties are similar to those that we have studied earlier

2.2 Addition of Rational Numbers

We have seen that if x and y are natural numbers, then $x + y$ is also a natural number. Likewise, if x and y are whole numbers, then so is their sum $x + y$ We have also seen that the sum of two integers is an integer. Here, we define the sum of two rational numbers. We shall define the sum in such a way that, like the sum of integers, the sum of two rational numbers is a rational number.

Suppose x and y are two rational numbers having the same denominator, say $x = \frac{-18}{11}$ and $y = \frac{6}{11}$. Then we obtain their sum as

$$x + y = \frac{-18}{11} + \frac{6}{11} = \frac{-18+6}{11} = \frac{-12}{11},$$

which is a rational number whose numerator is the sum of the numerators – 18 and 6 and whose denominator 11 is the common denominator of x and y.
Similarly,

$$\frac{4}{5} + \frac{13}{5} = \frac{4+13}{5} = \frac{17}{5},$$

$$\frac{6}{23} + \frac{-17}{23} = \frac{6+(-17)}{23} = \frac{-11}{23},$$

and $$\frac{-11}{29}+\frac{-25}{29}=\frac{-11+(-25)}{29}=\frac{-36}{29}$$

This rule of addition, therefore, may be stated as

$$\boxed{\frac{p}{q}+\frac{r}{q}=\frac{p+r}{q}}$$

How to add x and y, if they have different denominators ? Suppose $x=\frac{8}{7}$ and $y=\frac{15}{11}$. First we replace x and y by equivalent rational numbers with common denominator.

Here, $$x=\frac{8}{7}=\frac{8\times 11}{7\times 11}=\frac{88}{77}$$

$$y=\frac{15}{11}=\frac{15\times 7}{11\times 7}=\frac{105}{77}$$

Thus, $$x+y=\frac{8}{7}+\frac{15}{11}=\frac{88}{77}+\frac{105}{77}=\frac{193}{77}$$

Example 1 : Find the sum of

(i) $\frac{-2}{5}$ and $\frac{3}{8}$ (ii) $\frac{11}{25}$ and $\frac{19}{36}$

Solution : (i) We first find equivalent rational numbers with common denominator 5×8.

$$\frac{-2}{5}=\frac{-2\times 8}{5\times 8}=\frac{-16}{40}$$

$$\frac{3}{8}=\frac{3\times 5}{8\times 5}=\frac{15}{40}$$

Therefore,

$$\frac{-2}{5}+\frac{3}{8}=\frac{-16}{40}+\frac{15}{40}$$

$$=\frac{-16+15}{40}=\frac{-1}{40}$$

(ii) Here, the common denominator is 25 × 36.

$$\frac{11}{25}=\frac{11\times36}{25\times36}=\frac{396}{900}$$

$$\frac{19}{36}=\frac{19\times25}{36\times25}=\frac{475}{900}$$

Thus, $\frac{11}{25}+\frac{19}{36}=\frac{396}{900}+\frac{475}{900}=\frac{396+475}{900}=\frac{871}{900}$

In general, if $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers, then

$$\frac{p}{q}=\frac{p\times s}{q\times s} \text{ and } \frac{r}{s}=\frac{r\times q}{s\times q}$$

Therefore, $\frac{p}{q}+\frac{r}{s}=\frac{p\times s}{q\times s}+\frac{r\times q}{s\times q}$

$$=\frac{p\times s+r\times q}{q\times s}$$

We thus have the following working rule :

$$\boxed{\frac{p}{q}+\frac{r}{s}=\frac{p\times s+r\times q}{q\times s}}$$

Example 2 : Find the sum of

(i) $\frac{5}{18}$ and $\frac{3}{13}$ (ii) $\frac{-4}{11}$ and $\frac{5}{14}$

Solution : (i) $\frac{5}{18}+\frac{3}{13}=\frac{(5\times13)+(3\times18)}{18\times13}$

$$=\frac{65+54}{234}$$

$$=\frac{119}{234}$$

(ii) $$\frac{-4}{11}+\frac{5}{14}=\frac{(-4\times14)+(5\times11)}{11\times14}$$

$$=\frac{-56+55}{154}$$

$$=\frac{-1}{154}$$

As remarked in the previous Chapter, if the two denominators have a common factor, we may take the common denominator as the LCM of the two denominators instead of their product.

Example 3 : Add

(i) $\frac{-9}{10}$ and $\frac{22}{15}$ (ii) $\frac{33}{18}$ and $\frac{17}{26}$

Solution : (i) Here, LCM of 10 and 15 = 30.

Therefore, $$\frac{-9}{10}=\frac{-9\times3}{10\times3}=\frac{-27}{30}$$

$$\frac{22}{15}=\frac{22\times2}{15\times2}=\frac{44}{30}$$

Therefore, $$\frac{-9}{10}+\frac{22}{15}=\frac{-27}{30}+\frac{44}{30}=\frac{17}{30}$$

(ii) Here, LCM of 18 and 26 = 234

Therefore, $$\frac{33}{18}=\frac{33\times13}{18\times13}=\frac{429}{234}$$

$$\frac{17}{26}=\frac{17\times9}{26\times9}=\frac{153}{234}$$

Therefore, $$\frac{33}{18}+\frac{17}{26}=\frac{429}{234}+\frac{153}{234}$$

$$=\frac{582}{234}$$

2.3 Properties of Addition

We know that if x and y are two natural numbers, whole numbers or integers, then

$$x + y = y + x$$

Do we have the same property, if x and y are rational numbers ? Let us take a few examples .

(i) $$\frac{-9}{13}+\frac{17}{13}=\frac{-9+17}{13}=\frac{8}{13},$$

and $$\frac{17}{13}+\frac{-9}{13}=\frac{17+(-9)}{13}=\frac{8}{13}$$

(ii) $$\frac{-9}{5}+\frac{-4}{7}=\frac{-9\times7+(-4)\times5}{5\times7}=\frac{-83}{35}$$

$$\frac{-4}{7}+\frac{-9}{5}=\frac{-4\times5+(-9)\times7}{7\times5}=\frac{-83}{35}$$

(iii) $$\frac{8}{15}+\frac{7}{10}=\frac{8\times2+7\times3}{30}=\frac{37}{30}$$

$$\frac{7}{10}+\frac{8}{15}=\frac{7\times3+8\times2}{30}=\frac{37}{30}$$

In all these cases, we observe that

$$x + y = y + x$$

In fact, we have ·

Property I : *If x and y are two rational numbers, then*

$$x + y = y + x$$

We know how to find the sum of two rational numbers. What about the sum of three rational numbers? Like integers, for rational numbers also, we first take the sum of two rational numbers, and then add to the sum, the third rational number For example, to find the sum of $\frac{-5}{4}, \frac{9}{4}$ and $\frac{13}{4}$, we proceed as follows:

$$\frac{-5}{4}+\frac{9}{4}+\frac{13}{4}=\left(\frac{-5}{4}+\frac{9}{4}\right)+\frac{13}{4}$$

$$= \frac{4}{4} + \frac{13}{4}$$

$$= \frac{17}{4}$$

Like integers, we could add these numbers as

$$\frac{-5}{4} + \left(\frac{9}{4} + \frac{13}{4}\right) = \frac{-5}{4} + \frac{22}{4} = \frac{17}{4},$$

and we get the same rational number as earlier. Take one more example.

$$\frac{3}{2} + \frac{-5}{3} + \frac{4}{5} = \left(\frac{3}{2} + \frac{-5}{3}\right) + \frac{4}{5}$$

$$= \frac{3 \times 3 + (-5) \times 2}{2 \times 3} + \frac{4}{5}$$

$$= \frac{-1}{6} + \frac{4}{5}$$

$$= \frac{-1 \times 5 + 4 \times 6}{6 \times 5}$$

$$= \frac{19}{30}$$

and $$\frac{3}{2} + \left(\frac{-5}{3} + \frac{4}{5}\right) = \frac{3}{2} + \frac{-5 \times 5 + 4 \times 3}{3 \times 5}$$

$$= \frac{3}{2} + \frac{-13}{15}$$

$$= \frac{3 \times 15 + (-13) \times 2}{2 \times 15}$$

$$= \frac{19}{30}$$

We observe in these two examples that while adding three rational numbers, the way we pair them for addition does not alter the final sum. In fact, we have :

Property II : *If x, y and z are three rational numbers, then*

$$(x + y) + z = x + (y + z)$$

If we have four rational numbers x, y, z and t for addition, then we may add them in several ways .

$$\begin{aligned}(x + y) + (z + t) &= [(x + y) + z] + t \\ &= [x + (y + z)] + t \\ &= x + [(y + z) + t] \\ &= x + [y + (z + t)]\end{aligned}$$

We write the common value as $x + y + z + t$

Example 4 : Find the sum of

(i) $\frac{27}{13}, \frac{25}{13}, \frac{-19}{13}$ and $\frac{-21}{13}$.

(ii) $\frac{2}{3}, \frac{-3}{5}, \frac{1}{6}$ and $\frac{-8}{15}$.

Solution : (i) $$\begin{aligned}\frac{27}{13}+\frac{25}{13}+\frac{-19}{13}+\frac{-21}{13} &= \left(\frac{27}{13}+\frac{25}{13}\right)+\left(\frac{-19}{13}+\frac{-21}{13}\right) \\ &= \frac{52}{13}+\frac{-40}{13} \\ &= \frac{12}{13}\end{aligned}$$

(ii) $$\begin{aligned}\frac{2}{3}+\frac{-3}{5}+\frac{1}{6}+\frac{-8}{15} &= \left(\frac{2}{3}+\frac{-3}{5}\right)+\left(\frac{1}{6}+\frac{-8}{15}\right) \\ &= \frac{2\times5+(-3)\times3}{3\times5}+\frac{1\times5+(-8)\times2}{30} \\ &= \frac{1}{15}+\frac{-11}{30}\end{aligned}$$

$$= \frac{1\times 2+(-11)\times 1}{30}$$

$$= \frac{-9}{30}$$

$$= \frac{-3}{10}$$

As a consequence of Properties I and II, to find the sum of three or more rational numbers, we may arrange them in any order we like. The arrangement does not alter the sum. For example, Example 4 may also be solved as follows by arranging the terms in a different order :

(i) $$\frac{27}{13}+\frac{25}{13}+\frac{-19}{13}+\frac{-21}{13} = \left(\frac{27}{13}+\frac{-21}{13}\right)+\left(\frac{25}{13}+\frac{-19}{13}\right)$$

$$= \frac{6}{13}+\frac{6}{13}$$

$$= \frac{12}{13}$$

(ii) $$\frac{2}{3}+\frac{-3}{5}+\frac{1}{6}+\frac{-8}{15} = \left(\frac{2}{3}+\frac{1}{6}\right)+\left(\frac{-3}{5}+\frac{-8}{15}\right)$$

$$= \left(\frac{2\times 2+1}{6}\right)+\frac{-3\times 3+(-8)}{15}$$

$$= \frac{5}{6}+\frac{-17}{15}$$

$$= \frac{5\times 5+(-17)\times 2}{30}$$

$$= \frac{-9}{30}$$

$$= \frac{-3}{10}$$

We know that zero (0) as an integer has a peculiar property. If we add it to an integer, the sum is the integer itself. Since 0 is also a rational number, let us see what happens when we add it to a rational number

$$0+\frac{7}{9}=\frac{0}{9}+\frac{7}{9}=\frac{0+7}{9}=\frac{7}{9}$$

(Recall that 0 as a rational number can be written in an equivalent form $\frac{0}{q}$, where q is any non-zero integer.)

Similarly,

$$\frac{-15}{209}+0=\frac{-15}{209}+\frac{0}{209}=\frac{-15+0}{209}=\frac{-15}{209}$$

$$\frac{p}{q}+0=\frac{p}{q}+\frac{0}{q}=\frac{p+0}{q}=\frac{p}{q}$$

In fact, we have .

Property III : *If x is a rational number, then* $0 + x = x + 0 = x$.

We know that for any integer x,

$$x + (-x) = 0$$

For example, $5 + (-5) = 0$, $-101 + 101 = 0$

The integer $-x$ is called the *negative* of x,

Thus, -5 is the negative of 5 and 101 is the negative of -101. For rational numbers too, we have a similar result.

If $x=\frac{p}{q}$ is a rational number, then $\frac{-p}{q}$ is a rational number such that

$$\frac{p}{q}+\left(\frac{-p}{q}\right)=\frac{0}{q}=0$$

In other words, $\frac{-p}{q}$ is the negative of the rational number $x=\frac{p}{q}$. For example, $\frac{-3}{7}$ is the negative of $\frac{3}{7}$ and $\frac{9}{11}$ is the negative of $\frac{-9}{11}$. We thus have :

Property IV : *If x is a rational number, then $-x$ is a rational number such that* $x + (-x) = (-x) + x = 0$. $-x$ is called the *negative* of x.

It follows from here that if $-x$ is the *negative* of x, then x is the *negative* of $-x$, i.e.,

$$x = -(-x).$$

Example 5 : (i) For $x = \frac{8}{5}$ and $x = \frac{-5}{7}$, verify that $-(-x) = x$.

(ii) For $x = \frac{2}{3}, y = \frac{5}{7}$ and $x = \frac{-2}{3}, y = \frac{-5}{7}$,

verify that $-(x+y) = (-x) + (-y)$.

Solution : (i) $-\left(-\frac{8}{5}\right) = -\left(\frac{-8}{5}\right) = \frac{-(-8)}{5} = \frac{8}{5}$,

and $-\left(-\left(\frac{-5}{7}\right)\right) = -\left(\frac{-(-5)}{7}\right) = -\left(\frac{5}{7}\right) = \frac{-5}{7}$

Thus, in both the cases, $-(-x) = x$.

(ii) For $x = \frac{2}{3}$ and $y = \frac{5}{7}$, $-(x+y) = -\left(\frac{2}{3} + \frac{5}{7}\right) = -\left(\frac{14+15}{21}\right) = \frac{-29}{21}$

and $(-x) + (-y) = \left(-\left(\frac{2}{3}\right)\right) + \left(-\left(\frac{5}{7}\right)\right) = \left(\frac{-2}{3}\right) + \left(\frac{-5}{7}\right) = \frac{-2\times7+(-5)\times3}{21} = \frac{-29}{21}$

For $x = \frac{-2}{3}$ and $y = \frac{-5}{7}$,

$$-(x+y) = -\left(\frac{-2}{3} + \frac{-5}{7}\right)$$

$$= -\frac{-2\times7+(-5)\times3}{3\times7} = -\frac{-29}{21} = \frac{-(-29)}{21}$$

$$= \frac{29}{21}$$

and $$(-x)+(-y)=\left(-\left(\frac{-2}{3}\right)\right)+\left(-\left(\frac{-5}{7}\right)\right)$$

$$=\frac{-(-2)}{3}+\frac{-(-5)}{7}$$

$$=\frac{2}{3}+\frac{5}{7}$$

$$=\frac{2\times7+5\times3}{3\times7}$$

$$=\frac{29}{21}$$

Thus, in both the cases,

$$-(x+y)=(-x)+(-y)$$

EXERCISE 2.1

1. Add :

(i) $\frac{6}{7}$ and $\frac{4}{7}$ (ii) $\frac{7}{13}$ and $\frac{-6}{13}$

(iii) $\frac{6}{17}$ and $\frac{11}{-17}$ (iv) $\frac{-23}{28}$ and $\frac{5}{-28}$

2. Simplify :

(i) $\frac{-7}{9}+\frac{3}{4}$ (ii) $\frac{-3}{-11}+\frac{5}{9}$

(iii) $\frac{-8}{19}+\frac{-2}{57}$ (iv) $\frac{-7}{26}+\frac{-11}{39}$

3. Verify :

(i) $\frac{-5}{11}+\frac{-6}{13}=\frac{-6}{13}+\frac{-5}{11}$ (ii) $\frac{-7}{9}+(-4)=-4+\frac{-7}{9}$

(iii) $\frac{-8}{9}+(-7)=-7+\left(\frac{-8}{9}\right)$ (iv) $\frac{4}{11}+\frac{-5}{8}=\frac{-5}{8}+\frac{4}{11}$

4. Verify :

(i) $\frac{3}{4}+\left(\frac{-5}{6}+\frac{7}{8}\right)=\left(\frac{3}{4}+\frac{-5}{6}\right)+\frac{7}{8}$ (ii) $\frac{-1}{3}+\left(\frac{4}{9}+\frac{-8}{13}\right)=\left(\frac{-1}{3}+\frac{4}{9}\right)+\frac{-8}{13}$

(iii) $\frac{-2}{3}+\left(\frac{7}{8}+\frac{-3}{5}\right)=\left(\frac{-2}{3}+\frac{7}{8}\right)+\frac{-3}{5}$ (iv) $\frac{-3}{4}+\left(\frac{2}{5}+\frac{-4}{7}\right)=\left(\frac{-3}{4}+\frac{2}{5}\right)+\frac{-4}{7}$

5. Simplify the following :

(i) $\frac{2}{5}+\frac{8}{3}+\frac{-11}{15}+\frac{4}{5}+\frac{-2}{3}$ (ii) $\frac{4}{7}+0+\frac{-8}{9}+\frac{-13}{7}+\frac{17}{21}$

2.4 Subtraction of Rational Numbers

Let us recall that if x and y are two integers, then subtracting y from x is the same as adding $-y$ to x, i.e., $x - y = x + (-y)$.

Here, symbol '–' on the left side denotes the operation of subtraction, while '–' on the right hand side denotes negative of y. We define the subtraction of two rational numbers, in the same way. Specifically, if $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers, then

$$\boxed{\frac{p}{q}-\frac{r}{s}=\frac{p}{q}+\left(-\frac{r}{s}\right)}$$

Thus,

$$\frac{2}{3}-\frac{1}{5}=\frac{2}{3}+\left(-\frac{1}{5}\right)$$

$$\frac{6}{11}-\left(\frac{-8}{7}\right)=\frac{6}{11}+\left(-\left(\frac{-8}{7}\right)\right)=\frac{6}{11}+\frac{8}{7}$$

$$\frac{-13}{15}-\frac{19}{27}=\frac{-13}{15}+\left(\frac{-19}{27}\right)$$

$$\frac{-8}{21}-\left(\frac{-15}{19}\right)=\frac{-8}{21}+\left(-\left(\frac{-15}{19}\right)\right)=\frac{-8}{21}+\frac{15}{19}$$

The expressions on the right hand side can be simplified using the rules of addition

Example 6 : Subtract

(i) $\frac{1}{5}$ from $\frac{2}{3}$ (ii) $\frac{-8}{7}$ from $\frac{6}{11}$

(iii) $\frac{19}{27}$ from $\frac{-13}{15}$ (iv) $\frac{-15}{19}$ from $\frac{-8}{21}$

Solution :

(i) $$\frac{2}{3}-\frac{1}{5}=\frac{2}{3}+\left(\frac{-1}{5}\right)$$

$$=\frac{2\times5+(-1)\times3}{15}$$

$$=\frac{7}{15}$$

(ii) $$\frac{6}{11}-\left(\frac{-8}{7}\right)=\frac{6}{11}+\left(-\left(\frac{-8}{7}\right)\right)$$

$$=\frac{6}{11}+\frac{8}{7}$$

$$=\frac{6\times7+8\times11}{11\times7}$$

$$=\frac{130}{77}$$

iii) $$\frac{-13}{15}-\frac{19}{27}=\frac{-13}{15}+\left(\frac{-19}{27}\right)$$

$$=\frac{-13\times9+(-19)\times5}{135}$$

$$=\frac{-212}{135}$$

(iv) $$\frac{-8}{21}-\left(\frac{-15}{19}\right)=\frac{-8}{21}+\left(-\left(\frac{-15}{19}\right)\right)$$

$$=\frac{-8}{21}+\frac{15}{19}$$

$$=\frac{-8\times19+15\times21}{21\times19}$$

$$=\frac{163}{399}$$

Example 7: What should be added to

(i) $\frac{19}{27}$ to make it $\frac{-13}{15}$? (ii) $\frac{-15}{19}$ to make it $\frac{-8}{21}$?

Solution: We shall solve (i) and (ii) by two alternative methods.

(i) Suppose we add $\frac{p}{q}$ to $\frac{19}{27}$ to make it $\frac{-13}{15}$.

Then $$\frac{19}{27}+\frac{p}{q}=\frac{-13}{15}$$

Adding $\frac{-19}{27}$ to both sides, we get

$$\frac{19}{27}+\frac{p}{q}+\frac{-19}{27}=\frac{-13}{15}+\left(\frac{-19}{27}\right)$$

i.e., $$\frac{p}{q}=\frac{-13}{15}+\frac{-19}{27}$$

$$=\frac{-13\times9+(-19)\times5}{135}$$

$$=\frac{-212}{135}=-\frac{212}{135}$$

(ii) The required number is $\frac{-8}{21}-\left(\frac{-15}{19}\right) = \frac{-8}{21}-\left(-\left(\frac{15}{19}\right)\right)$

$$= \frac{-8}{21}+\frac{15}{19}$$

$$= \frac{-8\times19+15\times21}{21\times19}$$

$$= \frac{-152+315}{399}$$

$$= \frac{163}{399}$$

Check : $\frac{-15}{19}+\frac{163}{399}=\frac{-15\times21+163}{399}=\frac{-315+163}{399}=\frac{-152}{399}=\frac{-8}{21}$ (lowest form)

We know that if x and y are integers, then $x - y \neq y - x$. Likewise, for rational numbers too, we may check that $x-y \neq y-x$. As an illustration, we take $x=\frac{3}{5}$ and $y=\frac{7}{9}$. Then

$$x-y=\frac{3}{5}-\frac{7}{9}=\frac{3\times9-7\times5}{5\times9}=\frac{-8}{45}=-\frac{8}{45}, \text{ and}$$

$$y-x=\frac{7}{9}-\frac{3}{5}=\frac{7\times5-3\times9}{5\times9}=\frac{8}{45}$$

Thus, we have $x - y \neq y - x$. Note that $x - y = -(y - x)$

Further, we know that for integers x, y and z,

$(x + y) + z = x + (y + z)$, but $(x - y) - z = x - (y - z)$ is not true unless $z = 0$. The same is true for the subtraction of rational numbers too. For example, if $x=\frac{1}{2}, y=\frac{1}{3}$ and $z=\frac{1}{7}$, then

$$(x-y)-z=\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{1}{7}=\frac{1}{6}-\frac{1}{7}=\frac{1}{42},$$

but $$x-(y-z)=\frac{1}{2}-\left(\frac{1}{3}-\frac{1}{7}\right)=\frac{1}{2}-\frac{4}{21}=\frac{13}{42}.$$

However, as regards zero, for all rational numbers x, we do have

$$x-0=x \text{ (like } x+0=x)$$

but we do not have

$$0-x=x, \text{ (even though } 0+x=x)$$

unless x is zero.

2.5 A Working Rule for Simplification

Suppose we wish to simplify the expression

$$\frac{3}{8}+\frac{1}{2}+\left(\frac{-2}{5}\right)+\frac{7}{8}-\frac{3}{2}+\frac{4}{5}.$$

We may simplify it by taking two numbers at a time. However, we may simplify such expressions more easily by using the following steps :

Step I: Find the LCM of the denominators of all the numbers involved (In our case, this LCM is 40). This LCM is the denominator of the required rational number obtained after simplification (Which may not be in the lowest form as yet).

Step II: To find the numerator of the simplified rational number, we do the following :

(i) Divide the LCM by the denominator of the first rational number and get a quotient (In our case, it is 5).

(ii) Multiply the first numerator by the quotient obtained above and get an integer (In our case, it is 15).

(iii) Repeat steps (i) and (ii) for all rational numbers (In our case, corresponding integers are 20, – 16, 35, 60 and 32).

(iv) Retain the given symbols of addition and subtraction between the given rationals and get an expression involving integers (In our case, the expression is 15 + 20 + (– 16) + 35 – 60 + 32).

(v) Simplify the expression in (iv) to get the numerator of the simplified number (In our case, the numerator becomes 26).

Step III: Reduce the obtained number to the lowest form if not already so This is the simplied number (In our case, we reduce $\frac{26}{40}$ and get $\frac{13}{20}$ in the lowest form).

Example 8: Express as a rational number :

$$\frac{2}{5}+\frac{8}{3}-\frac{12}{15}+\frac{4}{5}-\frac{2}{3}$$

Solution: Required denominator = LCM of 5, 3, 15, 5, 3 = 15 (Step I)

$\frac{2}{5}$: $15 \div 5 = 3$ [Step II (i)]

$2 \times 3 = \boxed{6}$ [Step II (ii)]

$\frac{8}{3}$: $15 \div 3 = 5$

$8 \times 5 = \boxed{40}$ [Step II (iii)]

$\frac{12}{15}$: $15 \div 15 = 1$

$12 \times 1 = \boxed{12}$ [Step II (iii)]

$\frac{4}{5}$: $15 \div 5 = 3$

$4 \times 3 = \boxed{12}$ [Step II (iii)]

$\frac{2}{3}$: $15 \div 3 = 5$

$2 \times 5 = \boxed{10}$ [Step II (iii)]

Required numerator $= 6 + 40 - 12 + 12 - 10$
$= 58 - 22$
$= 36$

Thus, the required number $= \frac{36}{15} = \frac{12}{5}$ in lowest form (Step III).

The above procedure, in shorter form, may be written as follows :

$$\frac{2}{5}+\frac{8}{3}-\frac{12}{15}+\frac{4}{5}-\frac{2}{3}$$

$$=\frac{(2\times3)+(8\times5)-(12\times1)+(4\times3)-(2\times5)}{15}$$

$$=\frac{6+40-12+12-10}{15}$$

$$=\frac{58-22}{15}$$

$$=\frac{36}{15}=\frac{12}{5}$$

EXERCISE 2.2

1. Find the difference :

   (i) $\frac{13}{15}-\frac{12}{25}$ (ii) $\frac{7}{24}-\frac{19}{36}$

   (iii) $\frac{5}{63}-\frac{-8}{21}$ (iv) $\frac{-6}{13}-\frac{-7}{15}$

2. In each of the following, subtract the first number from the second, and the second number from the first. Examine in each case if the two answers are same.

   (i) $\frac{7}{8},\frac{5}{8}$ (ii) $\frac{1}{4},\frac{-1}{8}$ (iii) $\frac{8}{33},\frac{5}{22}$

3. The sum of two rational numbers is -8. If one of the numbers is $\frac{-15}{7}$, find the other.

4. The sum of two rational numbers is $\frac{1}{2}$. If one of the numbers is $\frac{-8}{19}$, find the other.

5. What number should be added to $\frac{-7}{8}$ so as to get $\frac{5}{9}$?

6. What number should be subtracted from $\frac{26}{33}$ so as to get $\frac{-5}{11}$?

7. Find the rational numbers in each pair and examine if they are equal :

(i) $\left(\frac{-8}{9}-\frac{11}{4}\right)-\frac{-4}{12}, \frac{-8}{9}-\left(\frac{11}{4}-\frac{-4}{12}\right)$

(ii) $\left(\frac{5}{14}-\frac{-7}{9}\right)-\frac{3}{42}, \frac{5}{14}-\left(\frac{-7}{9}-\frac{3}{42}\right)$

8. Simplify the following.

(i) $\frac{-2}{3}+\frac{5}{9}-\frac{-7}{6}$ (ii) $\frac{3}{8}-\frac{-2}{9}+\frac{-1}{36}$

(iii) $\frac{1}{6}+\frac{-2}{5}-\frac{-2}{15}$ (iv) $\frac{1}{12}+\frac{-5}{18}-\frac{7}{24}$

9. Fill in the blanks :

(i) $\frac{-4}{13}-\frac{-3}{26}=$ .... (ii) $\frac{-5}{14}+$ .... $=-1$

(iii) $\frac{-7}{9}+$ .... $=3$ (iv) .... $+\frac{15}{23}=4$

10. Write true (T) or false (F) for the following statements :

(i) $-4<\frac{-12}{5}-\frac{9}{5}$ (ii) $\frac{-3}{4}-\frac{-7}{5}>\frac{1}{5}$

(iii) The negative of a negative rational number is a positive rational number.

(iv) If x and y are two rational numbers such that $x>y$, then $x-y$ is always a positive rational number

2.6 Multiplication of Rational Numbers

The product or multiplication of two rational numbers is just like the product of two fractions. If $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers, then

$$\frac{p}{q}\times\frac{r}{s}=\frac{p\times r}{q\times s}=\frac{\text{product of the two numerators}}{\text{product of the two denominators}}$$

In other words, the product of two rational numbers is a rational number whose numerator is the product of the two numerators and whose denominator is the product of the two denominators.

Example 9: Find the product of

(i) $\frac{3}{8}$ and $\frac{-5}{7}$ (ii) $\frac{-2}{11}$ and $\frac{6}{5}$ (iii) $\frac{-3}{19}$ and $\frac{-5}{13}$

Solution: (i) $\frac{3}{8}\times\frac{-5}{7}=\frac{3\times(-5)}{8\times7}$

$$=\frac{-15}{56}$$

(ii) $\frac{-2}{11}\times\frac{6}{5}=\frac{(-2)\times6}{11\times5}$

$$=\frac{-12}{55}$$

(iii) $\frac{-3}{19}\times\frac{-5}{13}=\frac{(-3)\times(-5)}{19\times13}$

$$=\frac{15}{247}$$

2.7 Properties of Multiplication

We have studied some properties of multiplication of integers. The multiplication of rational numbers has similar properties. For example, if x and y are two integers, then we have

$$x\times y=y\times x$$

What would happen, if we take x and y as rational numbers? Take a few examples:

If $x=\frac{35}{9}$ and $y=\frac{7}{82}$, then

$$x\times y=\frac{35}{9}\times\frac{7}{82}=\frac{35\times7}{9\times82}$$

$$= \frac{245}{738}$$

and $$y \times x = \frac{7}{82} \times \frac{35}{9} = \frac{7 \times 35}{82 \times 9}$$

$$= \frac{245}{738}$$

Thus, $\frac{35}{9} \times \frac{7}{82} = \frac{7}{82} \times \frac{35}{9}$, i.e., $x \times y = y \times x$.

Similarly, $$\frac{21}{16} \times \frac{-11}{5} = \frac{21 \times (-11)}{16 \times 5} = \frac{-231}{80}$$

and $$\frac{-11}{5} \times \frac{21}{16} = \frac{-11 \times 21}{5 \times 16} = \frac{-231}{80}$$

Thus, here also $\frac{21}{16} \times \frac{-11}{5} = \frac{-11}{5} \times \frac{21}{16}$

In general, for $x = \frac{p}{q}$ and $y = \frac{r}{s}$,

$$x \times y = \frac{p}{q} \times \frac{r}{s} = \frac{p \times r}{q \times s},$$

$$y \times x = \frac{r}{s} \times \frac{p}{q} = \frac{r \times p}{s \times q}$$

We know, from the properties of multiplication of integers, that $p \times r = r \times p$, and $q \times s = s \times q$, and it gives

$$\frac{p}{q} \times \frac{r}{s} = \frac{r}{s} \times \frac{p}{q}, \text{ i.e., } x \times y = y \times x.$$

Thus, we have .

Property I: *If x and y are rational numbers, then $x \times y = y \times x$.*

Next, we know that if x, y and z are three integers, then

$$(x \times y) \times z = x \times (y \times z)$$

Let us see whether this property holds for rational numbers too. For example, if $x=\frac{-3}{4}$, $y=\frac{-2}{11}$ and $z=\frac{5}{7}$, then

$$(x \times y) \times z = \left(\frac{-3}{4}\times\frac{-2}{11}\right)\times\frac{5}{7} = \left(\frac{-3\times(-2)}{4\times 11}\right)\times\frac{5}{7}$$

$$= \frac{6}{44}\times\frac{5}{7}$$

$$= \frac{15}{154}$$

$$x \times (y \times z) = \frac{-3}{4}\times\left(\frac{-2}{11}\times\frac{5}{7}\right) = \frac{-3}{4}\times\left(\frac{-2\times 5}{11\times 7}\right)$$

$$= \frac{-3}{4}\times\frac{-10}{77}$$

$$= \frac{15}{154}$$

Thus, $(x \times y) \times z = x \times (y \times z)$

Likewise,

$$\left(\frac{4}{9}\times\frac{-13}{3}\right)\times\frac{15}{7} = \frac{4\times(-13)}{9\times 3}\times\frac{15}{7}$$

$$= \frac{-52}{27}\times\frac{15}{7}$$

$$= \frac{-780}{189} = \frac{-260}{63}$$

$$\frac{4}{9}\times\left(\frac{-13}{3}\times\frac{15}{7}\right) = \frac{4}{9}\times\left(\frac{-195}{21}\right)$$

$$= \frac{-780}{189} = \frac{-260}{63}$$

The above examples are special cases of the following property :

Property II: *If x, y and z are rational numbers, then*

$$(x \times y) \times z = x \times (y \times z)$$

As a consequence of this property for integers, we observed that we may multiply three or more integers without using brackets. Likewise, in multiplication of three or more rational numbers, we can ignore brackets. Taking suitable examples, we may verify that for rational numbers x, y, z and t,

$$\begin{aligned}(x \times y) \times (z \times t) &= [x \times (y \times z)] \times t = [(x \times y) \times z] \times t \\ &= x \times [y \times (z \times t)] = x \times [(y \times z) \times t]\end{aligned}$$

All these products may be written as $x \times y \times z \times t$.

We know that 0 and 1 are special integers, in the sense that multiplication by 0 makes every integer 0 while multiplication by 1 leaves the integer unchanged. Since 0 and 1 are rational numbers also, it is natural to expect that they behave similarly with multiplication of rational numbers.

Property III: If x is any *rational number, then* $x \times 0 = 0 = 0 \times x$.

As an illustration, we have

$$\begin{aligned}\frac{73}{82} \times 0 &= \frac{73}{82} \times \frac{0}{1} \\ &= \frac{73 \times 0}{82 \times 1} \\ &= \frac{0}{82} \\ &= 0\end{aligned}$$

$$0 \times \frac{-105}{279} = \frac{0}{1} \times \frac{-105}{279}$$

$$= \frac{0\times(-105)}{1\times 279}$$

$$= \frac{0}{279}$$

$$= 0$$

Property IV: *If x is a rational number, then* $x \times 1 = 1 \times x = x$

For example,

$$\frac{386}{273}\times 1 = \frac{386}{273}\times\frac{1}{1}$$

$$= \frac{386\times 1}{273\times 1} = \frac{386}{273}$$

$$1\times\frac{-983}{1010} = \frac{1}{1}\times\frac{-983}{1010}$$

$$= \frac{1\times(-983)}{1\times(1010)} = \frac{-983}{1010}$$

An important property of integers is that $x \times (y + z) = x \times y + x \times z$, i.e., first adding and then multiplying is the same as first multiplying and then adding.

Likewise, we have

$$x \times (y - z) = x \times y - x \times z$$

We have similar properties for rational numbers too.

Property V: *If x, y and z are rational numbers, then*

(i) $x \times (y + z) = x \times y + x \times z$.
(ii) $x \times (y - z) = x \times y - x \times z$.

Example 10: Find

$$\text{(i)}\quad \frac{2}{3}\times\left(\frac{9}{5}+\frac{6}{7}\right) \quad\text{and}\quad \text{(ii)}\quad \frac{2}{3}\times\frac{9}{5}+\frac{2}{3}\times\frac{6}{7},$$

and check that these are the same rational numbers.

Solution : (i)
$$\frac{2}{3}\times\left(\frac{9}{5}+\frac{6}{7}\right)=\frac{2}{3}\times\left(\frac{9\times7+5\times6}{5\times7}\right)$$
$$=\frac{2}{3}\times\frac{63+30}{35}$$
$$=\frac{2}{3}\times\frac{93}{35}$$
$$=\frac{186}{105}=\frac{62}{35}\ \text{(lowest form)}$$

(ii)
$$\frac{2}{3}\times\frac{9}{5}+\frac{2}{3}\times\frac{6}{7}=\frac{2\times9}{3\times5}+\frac{2\times6}{3\times7}$$
$$=\frac{18}{15}+\frac{12}{21}$$
$$=\frac{126+60}{105}$$
$$=\frac{186}{105}$$
$$=\frac{62}{35}\ \text{(lowest form)}$$

Hence, they are the same rational numbers.

Example 11: Simplify the following rational numbers and examine if they are equal:

$$\text{(i)}\quad \frac{7}{12}\times\left(\frac{28}{13}-\frac{5}{11}\right) \qquad \text{(ii)}\quad \frac{7}{12}\times\frac{28}{13}-\frac{7}{12}\times\frac{5}{11}$$

Solution :

(i) $\frac{7}{12}\times\left(\frac{28}{13}-\frac{5}{11}\right)$

$$= \frac{7}{12}\times\frac{28\times11-5\times13}{13\times11}$$

$$= \frac{7}{12}\times\frac{308-65}{143}$$

$$= \frac{7}{12}\times\frac{243}{143}$$

$$= \frac{1701}{1716}$$

(ii) $\frac{7}{12}\times\frac{28}{13}-\frac{7}{12}\times\frac{5}{11}$

$$= \frac{196}{156}-\frac{35}{132}$$

$$= \frac{196\times11-35\times13}{1716}$$

$$= \frac{2156-455}{1716}$$

$$= \frac{1701}{1716}$$

We observe that the two rational numbers are equal.

EXERCISE 2.3

1. Multiply:

(i) $\frac{3}{11}$ by $\frac{2}{5}$ (ii) $\frac{3}{7}$ by $\left(\frac{-2}{5}\right)$

(iii) $\left(\frac{-2}{9}\right)$ by $\frac{33}{54}$ (iv) $\left(\frac{-3}{7}\right)$ by $\frac{7}{5}$

(v) $\left(\frac{-5}{17}\right)$ by $\frac{3}{10}$ (vi) $\frac{6}{7}$ by $\left(\frac{-17}{18}\right)$

(vii) $\left(\frac{5}{-13}\right)$ by $\left(\frac{-26}{15}\right)$ (viii) $\left(\frac{-6}{11}\right)$ by $\frac{44}{13}$

(ix) $\left(\frac{9}{-11}\right)$ by $\left(\frac{22}{-27}\right)$ (x) $\left(\frac{-8}{25}\right)$ by $\left(\frac{-5}{16}\right)$

2. Simplify and express the result as a rational number in the standard form :

(i) $\frac{-8}{7}\times\frac{14}{5}$ (ii) $\frac{7}{3}\times\frac{-1}{28}$

(iii) $\frac{-14}{9}\times(-27)$ (iv) $\frac{13}{6}\times\frac{-18}{91}$

3. Verify the property $x \times y = y \times x$ by taking

(i) $x=\frac{-1}{5},\ y=\frac{2}{7}$ (ii) $x=\frac{2}{7},\ y=\frac{-11}{8}$

(iii) $x=0,\ y=\frac{-15}{4}$ (iv) $x=1,\ y=\frac{-7}{2}$

4. Fill in the blanks :

(i) $\frac{-5}{13}\times\frac{-6}{11}=\frac{-6}{11}\times\frac{-5}{13}$ (An illustration)

(ii) $-4\times\frac{5}{9}=$ ....

(iii) $\frac{3}{11}\times\frac{-5}{8}=$ .. ..

(iv) $-6\times\frac{-3}{-7}=$ ....

5. Verify the property $x \times (y \times z) = (x \times y) \times z$ by taking

(i) $x = \frac{7}{4}, y = \frac{-11}{3}, z = \frac{1}{2}$ (ii) $x = 0, y = \frac{-12}{5}, z = \frac{2}{5}$

(iii) $x = 1, y = \frac{-5}{2}, z = \frac{2}{5}$ (iv) $x = \frac{1}{2}, y = \frac{4}{3}, z = \frac{-7}{5}$

6. Fill in the blanks.

(i) $\left(\frac{-3}{4}\right) \times \left(\frac{4}{5} \times \frac{-7}{8}\right) = \left(\frac{-3}{4} \times \frac{4}{5}\right) \times \frac{-7}{8}$ (An illustration)

(ii) $\frac{1}{2} \times \left(\frac{3}{4} \times \frac{-5}{13}\right) =$ ....

(iii) $-4 \times \left(-6 \times \frac{-7}{11}\right) =$ ....

(iv) $\frac{-2}{9} \times \left(\frac{4}{5} \times \frac{3}{7}\right) =$ . ...

7. Verify the property $x \times (y + z) = (x \times y) + (x \times z)$ by taking

(i) $x = \frac{-3}{4}, y = \frac{5}{2}, z = \frac{7}{6}$ (ii) $x = -2, y = \frac{9}{5}, z = \frac{2}{3}$

(iii) $x = \frac{-5}{2}, y = \frac{16}{3}, z = -1$ (iv) $x = 0, y = \frac{-8}{3}, z = 1$

8. By taking values of x, y and z as in Question 7 above, verify that

$x \times (y - z) = x \times y - x \times z.$

9. Fill in the blanks :

(i) $\frac{-4}{5} \times \left(\frac{5}{7} + \frac{-8}{9}\right) = \left(\frac{-4}{5} \times \frac{5}{7}\right) + \left(\frac{-4}{5} \times \frac{-8}{9}\right)$ (An illustration)

(ii) $\frac{-3}{8} \times \left(\frac{-6}{11} + \frac{4}{9}\right) =$ ....

(iii) $6\times\left(\frac{5}{13}+\frac{-3}{4}\right)=$ ....

(iv) $\frac{2}{3}\times\left(\frac{-5}{7}-\frac{4}{5}\right)=$.. ....

(v) $\frac{-4}{7}\times\left(\frac{3}{4}-\frac{-2}{3}\right)=$ .. ..

10. Write true (T) or false (F) for the following statements :

(i) $-1\times x$ is positive, if x is negative.

(ii) $-1\times(0-x)$ is negative, if x is positive.

(iii) $-x\times y$ is positive, if y is negative

(iv) $x\times(y+z)$ is non-zero, if x is non-zero.

(v) If $x\times(y-z)$ is zero, then $y=z$.

(vi) Product of two rational numbers can never be an integer.

(vii) Product of two integers is never a fraction.

(viii) For a rational number x, $x\times y=y+y+\ldots x$ times.

2.8 Reciprocal of a Rational Number

Consider a rational number $\frac{3}{7}$ which is made up of two integers 3 as numerator and 7 as denominator. If we interchange the integers in numerator and denominator, we get another rational number $\frac{7}{3}$. In general for any non-zero rational number $\frac{p}{q}$, we have another non-zero rational number $\frac{q}{p}$. This number is called the *reciprocal* of $\frac{p}{q}$ The number $\frac{7}{3}$ is the reciprocal of $\frac{3}{7}$. Likewise, $\frac{9}{-13}$ or $\frac{-9}{13}$ is the reciprocal of $\frac{-13}{9}$, and $\frac{-105}{113}$ is the reciprocal of $\frac{113}{-105}$ or $\frac{-113}{105}$.

We observe from here that if $\frac{q}{p}$ is the reciprocal of $\frac{p}{q}$, then $\frac{p}{q}$ is the reciprocal of $\frac{q}{p}$. In other words, $\frac{p}{q}$ and $\frac{q}{p}$ are reciprocals of each other. If we denote the rational number $\frac{p}{q}$ by x, then its reciprocal $\frac{q}{p}$ is denoted by x^{-1}. Put in words, we say

$$\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}.$$

We observe two things from the above :

1. $\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$

Therefore, $\left(\left(\frac{p}{q}\right)^{-1}\right)^{-1} = \left(\frac{q}{p}\right)^{-1}$

But $\left(\frac{q}{p}\right)^{-1}$, the reciprocal of $\frac{q}{p}$, is $\frac{p}{q}$.

Therefore, $\left(\left(\frac{p}{q}\right)^{-1}\right)^{-1} = \frac{p}{q}$

i.e., for a rational number x,

$$(x^{-1})^{-1} = x.$$

In other words, the reciprocal of the reciprocal of a rational number is the number itself.

2. $\left(\frac{p}{q}\right) \times \left(\frac{p}{q}\right)^{-1} = \frac{p}{q} \times \frac{q}{p}$

$$= \frac{p \times q}{q \times p}$$

$$= 1, \text{ since } p \times q = q \times p$$

i.e., for a non-zero rational number x,

$$x \times x^{-1} = 1.$$

In other words, the product of a non-zero rational number and its reciprocal is always 1. Because of this relationship, the reciprocal x^{-1} of x is also denoted as $\frac{1}{x}$, i.e., $x^{-1} = \frac{1}{x}$.

We also observe that 0 as a rational number has no reciprocal, because $0 = \frac{0}{1}$ and $\frac{1}{0}$ is not a rational number

Example 12: Find the reciprocal of the rational numbers

(i) $\frac{3}{7} + \frac{5}{11}$ (ii) $\frac{3}{7} - \frac{5}{11}$ (iii) $\frac{3}{7} \times \frac{5}{11}$

(iv) $\left|\frac{-3}{7}\right|$ (v) $-\left|\frac{5}{11}\right|$

Solution : (i) $= \frac{3}{7} + \frac{5}{11} = \frac{3 \times 11 + 5 \times 7}{7 \times 11}$

$$= \frac{33 + 35}{77}$$

$$= \frac{68}{77}$$

Therefore, $\left(\frac{3}{7} + \frac{5}{11}\right)^{-1} = \left(\frac{68}{77}\right)^{-1}$

$$= \frac{77}{68}$$

(ii) $$\frac{3}{7}-\frac{5}{11}=\frac{3\times11-5\times7}{7\times11}$$

$$=\frac{33-35}{77}$$

$$=\frac{-2}{77}$$

Therefore, $$\left(\frac{3}{7}-\frac{5}{11}\right)^{-1}=\left(\frac{-2}{77}\right)^{-1}=\frac{-77}{2}$$

(iii) $$\frac{3}{7}\times\frac{5}{11}=\frac{3\times5}{7\times11}$$

$$=\frac{15}{77}$$

Therefore, $$\left(\frac{3}{7}\times\frac{5}{11}\right)^{-1}=\left(\frac{15}{77}\right)^{-1}=\frac{77}{15}$$

(iv) $$\left|\frac{-3}{7}\right|^{-1}=\left(\frac{3}{7}\right)^{-1}=\frac{7}{3}$$

(v) $$-\left|\frac{5}{11}\right|=\frac{-5}{11}$$

Therefore, $$\left(-\left|\frac{5}{11}\right|\right)^{-1}=\left(\frac{-5}{11}\right)^{-1}=\frac{-11}{5}$$

Example 13: If x and y are non-zero rational numbers, then write true (T) or false (F) for the following statements:

1. $(x+y)^{-1} = x^{-1}+y^{-1}$
2. $(x-y)^{-1} = x^{-1}-y^{-1}$
3. $(x\times y)^{-1} = x^{-1}\times y^{-1}$
4. $|x^{-1}| = |x|^{-1}$

Solution : In the above example, let $x=\frac{3}{7}$ and $y=\frac{5}{11}$.

We observe that

$$\left(\frac{3}{7}+\frac{5}{11}\right)^{-1}=\left(\frac{33+35}{77}\right)^{-1}=\left(\frac{68}{77}\right)^{-1}=\frac{77}{68},$$

but $\left(\frac{3}{7}\right)^{-1}+\left(\frac{5}{11}\right)^{-1}=\left(\frac{7}{3}\right)+\left(\frac{11}{5}\right)=\frac{35+33}{15}=\frac{68}{15}$.

Thus, the statement 1 above is false.

Further,

$$\left(\frac{3}{7}-\frac{5}{11}\right)^{-1}=\left(\frac{33-35}{77}\right)^{-1}=\frac{-77}{2},$$

but $\left(\frac{3}{7}\right)^{-1}-\left(\frac{5}{11}\right)^{-1}=\frac{7}{3}-\frac{11}{5}=\frac{2}{15}$

Therefore, the statement 2 is also false.

Next,

$$\left(\frac{3}{7}\times\frac{5}{11}\right)^{-1}=\left(\frac{15}{77}\right)^{-1}=\frac{77}{15},$$

and $\left(\frac{3}{7}\right)^{-1}\times\left(\frac{5}{11}\right)^{-1}=\frac{7}{3}\times\frac{11}{5}=\frac{77}{15}$

Thus, the statement 3 is true for these values of x and y. We may take other values of x and y and verify that the statement is true. In fact, for any two non-zero rational numbers $\frac{p}{q}$ and $\frac{r}{s}$,

$$\left(\frac{p}{q}\times\frac{r}{s}\right)^{-1}=\left(\frac{p\times r}{q\times s}\right)^{-1}=\frac{q\times s}{p\times r},$$

and $$\left(\frac{p}{q}\right)^{-1}\times\left(\frac{r}{s}\right)^{-1}=\frac{q}{p}\times\frac{s}{r}=\frac{q\times s}{p\times r}$$

Statement 4 is also true for any non-zero rational number. If $x=\frac{p}{q}$, then

$$\left|x^{-1}\right|=\left|\left(\frac{p}{q}\right)^{-1}\right|=\left|\frac{q}{p}\right|=\frac{|q|}{|p|},$$

and $$|x|^{-1}=\left|\frac{p}{q}\right|^{-1}=\left(\frac{|p|}{|q|}\right)^{-1}=\frac{|q|}{|p|}$$

2.9 Division of Rational Numbers

Consider two rational numbers x and y with $y \neq 0$. We wish to divide x by y. Since the product of two rational numbers is a rational number, it is natural to expect that division of two rational numbers too should give us a rational number. This is indeed true.

Consider two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$. As for multiplication, we define the division by finding the division of numerators and denominators separately. More specifically, we write

$$\boxed{\frac{p}{q}\div\frac{r}{s}=\frac{p\div r}{q\div s}}$$

If the divisions $p \div r$ and $q \div s$ give integer quotients, then the above operation produces a rational number. For example,

$$\frac{8}{15}\div\frac{2}{3}=\frac{8\div 2}{15\div 3}=\frac{4}{5}$$

$$\frac{81}{605}\div\frac{27}{11}=\frac{81\div 27}{605\div 11}=\frac{3}{55}$$

However, if one or both of $p \div r$ and $q \div s$ do not produce integer quotients, then $p \div r$ becomes a rational number $\frac{p}{r}$ and $q \div s$ is the rational number $\frac{q}{s}$. In this case, we get an expression of the type

$$\frac{\text{rational number}}{\text{rational number}},$$

and we end up with the problem of dividing a rational number with a rational number. We, therefore, adopt a different approach to define division in rational numbers. We use the idea of reciprocal of rational numbers to define division.

Let us take two integers, say 8 and 2. Then $8 \div 2 = 4$

Let us treat 8, 2 and 4 as rational numbers $\frac{8}{1}, \frac{2}{1}$ and $\frac{4}{1}$ respectively. Then the above relation becomes

$$\frac{8}{1} \div \frac{2}{1} = \frac{4}{1}$$

Also, $$\frac{8}{1} \times \frac{1}{2} = \frac{8 \times 1}{1 \times 2} = \frac{8}{2} = \frac{4}{1}$$

This can be written, using reciprocal, as

$$\frac{8}{1} \times \left(\frac{2}{1}\right)^{-1} = \frac{4}{1}$$

In other words, we have .

$$\frac{8}{1} \div \frac{2}{1} = \frac{8}{1} \times \left(\frac{2}{1}\right)^{-1}$$

If we write x, y and z for the rational numbers $\frac{8}{1}, \frac{2}{1}$ and $\frac{4}{1}$ respectively, then what we have got is a relation

$$x \div y = x \times y^{-1}$$

This gives the clue to the definition of division of a rational number by another rational number :

If x and y are rational numbers and $y \neq 0$, then

$$\boxed{x \div y = x \times y^{-1}}$$

Put into words, to divide a rational number x by another (non-zero) rational number y, we multiply x by the reciprocal of y.

Therefore,

$$8 \div \frac{5}{6} = \frac{8}{1} \times \frac{6}{5}$$

$$\frac{7}{3} \div \frac{11}{23} = \frac{7}{3} \times \frac{23}{11}$$

$$\frac{-5}{9} \div \frac{-8}{29} = \frac{-5}{9} \times \frac{-29}{8}, \text{ etc.}$$

Using properties of reciprocal of a rational number, we may verify that many properties shared by the multiplication operation no longer hold for the operation of division.

Property: *If x, y and z are rational numbers, then we have*

(i) $(x + y) \div z = x \div z + y \div z$

(ii) $(x - y) \div z = x \div z - y \div z$

However, none of the following statements is true :

(i) $x \div y = y \div x$

(ii) $(x \div y) \div z = x \div (y \div z)$

(iii) $x \div (y + z) = x \div y + x \div z$

(iv) $x \div (y - z) = x \div y - x \div z$

EXERCISE 2.4

1. Find the reciprocal of

(i) 17 (ii) -19 (iii) $\frac{8}{13}$ (iv) $\frac{-13}{29}$

2. Verify that

$$(x + y)^{-1} \neq x^{-1} + y^{-1}$$

by taking (i) $x = \frac{3}{5}, y = \frac{4}{9}$ (ii) $x = \frac{-7}{16}, y = \frac{-15}{8}$

3. Verify that

$$(x - y)^{-1} \neq x^{-1} - y^{-1}$$

by taking (i) $x = \frac{3}{5}, y = \frac{5}{7}$ (ii) $x = \frac{-7}{19}, y = \frac{-11}{13}$

4. Verify that

$$(x \times y)^{-1} = x^{-1} \times y^{-1}$$

by taking (i) $x = \frac{11}{23}, y = \frac{-17}{5}$ (ii) $x = \frac{19}{17}, y = \frac{-8}{31}$

5. Verify that

$$(x \div y)^{-1} = x^{-1} \div y^{-1}$$

by taking (i) $x = \frac{10}{11}, y = \frac{12}{23}$ (ii) $x = \frac{-15}{26}, y = \frac{5}{13}$

6. Find the value of

(i) $-4 \div \left(\frac{-3}{5}\right)$ (ii) $-\frac{1}{8} \div \frac{3}{4}$

(iii) $\left(\frac{-7}{12}\right) \div \left(\frac{-2}{3}\right)$ (iv) $\frac{3}{13} \div \left(\frac{-4}{65}\right)$

7. The product of two rational numbers is $\frac{-8}{9}$. If one of the numbers is $\frac{-4}{15}$, find the other

8. With what number should we multiply $\frac{-15}{28}$, so that the product be $\frac{-5}{7}$?

9. Taking $x = \frac{8}{15}, y = \frac{2}{5}$ and $z = \frac{4}{10}$, verify that $(x \div y) \times z \neq x \div (y \times z)$.

10. Write true (T) or false (F) for the following statements :

(i) $x \div (x \times x) = x$ for every non-zero rational number x.

(ii) $(x \div x) \times x = 1$ for every non-zero rational number x

(iii) $x \div (y + z) = x \div y + x \div z$ is true for $y = z$

(iv) $(x - y) \div z = x \div z - y \div z$ is true for all $z > 0$.

(v) For every non-zero rational number $x, -x \div x = x \div (-x)$.

2.10 Rational Numbers between Two Rational Numbers

Consider the integers 2 and 8. The integer $\frac{2+8}{2} = 5$ is an integer such that $2 < 5 < 8$, i.e., we have an integer between 2 and 8. However, if we take 5 and 6, then we know that there is no integer between 5 and 6. But the rational number $\frac{5+6}{2}=\frac{11}{2}$ does lie between 5 and 6. Thus, we see that between two given integers, there may or may not lie an integer, but there does always lie a rational number between two given integers. In fact, we have a much better result. It says that if x and y are two rational numbers, then there always lies a rational number between x and y. As a matter of fact, there lie hundreds and thousands of rational numbers between x and y. Actually, we can say much more. There are so many rational numbers between x and y that we cannot count them Out of so many rational numbers, one can be located easily. It is $\frac{x+y}{2}$.

If x and y are two rational numbers and $x < y$, then $(x + y) \div 2$ is a rational number such that

$$x < (x + y) \div 2 < y$$

For example, the rational number $\left(\frac{1}{2}+\frac{3}{4}\right)\div 2=\frac{5}{4}\div 2=\frac{5}{4}\times\frac{1}{2}=\frac{5}{8}$ is such that

$$\frac{1}{2}<\frac{5}{8}<\frac{3}{4}.$$

Similarly, $\left(\frac{1}{2}+\frac{5}{8}\right)\div 2=\frac{9}{8}\times\frac{1}{2}=\frac{9}{16}$ is a rational number such that

$$\frac{1}{2}<\frac{9}{16}<\frac{5}{8},$$

and $\left(\frac{5}{8}+\frac{3}{4}\right)\div 2=\frac{11}{8}\times\frac{1}{2}=\frac{11}{16}$ is a rational number such that

$$\frac{5}{8}<\frac{11}{16}<\frac{3}{4}.$$

Combining the above results, we get three rational numbers, viz., $\frac{9}{16}, \frac{5}{8}$ and $\frac{11}{16}$, all lying between $\frac{1}{2}$ and $\frac{3}{4}$. Needless to say, the process may be repeated as many times as we like and we can obtain as many rational numbers as we like between $\frac{1}{2}$ and $\frac{3}{4}$.

Having observed that *there are infinitely many rational numbers between any two given rational numbers,* we describe an *alternative* method of locating them. Suppose, we wish to find two rational numbers between $\frac{1}{3}$ and $\frac{3}{5}$. First we find equivalent rational numbers having a common denominator. Thus, we write

$$\frac{1}{3}=\frac{5}{15} \text{ and } \frac{3}{5}=\frac{9}{15}.$$

Now we choose two integers between the two numerators 5 and 9 of the equivalent rational numbers If we choose 6 and 7, then the rational numbers $\frac{6}{15}$ and $\frac{7}{15}$ are such that

$$\frac{5}{15}<\frac{6}{15}<\frac{7}{15}<\frac{9}{15},$$

i.e.,

$$\frac{1}{3}<\frac{2}{5}<\frac{7}{15}<\frac{3}{5}.$$

Between 5 and 9, there are only three integers. As such, we can find only three rational numbers between $\frac{1}{3}$ and $\frac{3}{5}$ using this method If we were to find, say, seven rational numbers between $\frac{1}{3}$ and $\frac{3}{5}$, we proceed as follows :

We replace the numbers by equivalent rational numbers having a sufficiently large common denominator.

$$\frac{1}{3}=\frac{5}{15}=\frac{10}{30}=\frac{100}{300}=\frac{10000}{30000}, \text{ etc. and}$$

$$\frac{3}{5}=\frac{9}{15}=\frac{18}{30}=\frac{180}{300}=\frac{18000}{30000}, \text{ etc.}$$

Now choosing $\frac{10}{30}$ for $\frac{1}{3}$ and $\frac{18}{30}$ for $\frac{3}{5}$ and finding seven integers 11, 12, 13, ..., 16, 17 between 10 and 18, we find seven rational numbers between $\frac{10}{30}$ and $\frac{18}{30}$, i.e.,

$$\frac{1}{3} < \frac{11}{30} < \frac{12}{30} < \dots\dots < \frac{16}{30} < \frac{17}{30} < \frac{3}{5}$$

To find fifty rational numbers between $\frac{1}{3}$ and $\frac{3}{5}$, we may take the equivalent rational numbers as $\frac{100}{300}$ and $\frac{180}{300}$, as there are sufficiently many integers between 100 and 180.

Example 14: (i) Find two rational numbers between $\frac{1}{5}$ and $\frac{1}{2}$.

(ii) Find seven rational numbers between $\frac{1}{3}$ and $\frac{1}{2}$.

Solution : (i) $\frac{1}{5} = \frac{2}{10}$ and $\frac{1}{2} = \frac{5}{10}$

Also, integers 3 and 4 lie between 2 and 5.

Therefore, $\frac{2}{10} < \frac{3}{10} < \frac{4}{10} < \frac{5}{10}$,

or $\frac{1}{5} < \frac{3}{10} < \frac{2}{5} < \frac{1}{2}$, i.e., $\frac{3}{10}$ and $\frac{2}{5}$ are two required rational numbers.

(ii) $\frac{1}{3} = \frac{16}{48}$ and $\frac{1}{2} = \frac{24}{48}$

Also, 17, 18, ..., 23 are seven integers between 16 and 24.

Therefore, $\frac{16}{48} < \frac{17}{48} < \frac{18}{48} < \dots < \frac{23}{48} < \frac{24}{48}$

or $\frac{1}{3} < \frac{17}{48} < \frac{3}{8} < \ldots < \frac{23}{48} < \frac{1}{2}$

i.e., $\frac{17}{48}, \frac{3}{8}, \ldots, \frac{23}{48}$ are seven required rational numbers.

EXERCISE 2.5

1. Find a rational number between

(i) -7 and -6 (ii) $\frac{-7}{6}$ and $\frac{-6}{7}$

(iii) $\frac{-5}{9}$ and $\left|\frac{-5}{9}\right|$ (iv) $\frac{-3}{5}$ and $\frac{|-3|}{5}$

2. Find three rational numbers between

(i) $\frac{7}{10}$ and $\frac{11}{10}$ (ii) $\frac{-10}{7}$ and $\frac{-10}{11}$

(iii) $\frac{28}{35}$ and $\frac{39}{15}$ (iv) $\frac{-21}{13}$ and $\frac{13}{21}$

3. For $x = \frac{3}{5}$ and $y = \frac{-3}{7}$, is $|x + y| = |x| + |y|$? If not, then find two rational numbers between $|x + y|$ and $|x| + |y|$.

4. For $x = \frac{-4}{9}$ and $y = \frac{5}{11}$, are $|x - y|$ and $|x| - |y|$ equal? If not, then insert two rational numbers between the two resulting numbers.

5. Insert five rational numbers between x and $|x|$, where

(i) $x = \frac{-9}{15}$ (ii) $x = \frac{-17}{20}$

6. Insert a rational number between $(x + y)^{-1}$ and $x^{-1} + y^{-1}$, where

(i) $x = \frac{3}{5}, y = \frac{7}{9}$ (ii) $x = \frac{-4}{7}, y = \frac{-11}{15}$

7. Insert a rational number between $(x-y)^{-1}$ and $x^{-1}-y^{-1}$, where

(i) $x=\frac{2}{3}$ and $y=\frac{3}{4}$ (ii) $x=\frac{-2}{3}$ and $y=\frac{-3}{4}$

8. For each pair of values in Question 7, determine if $(x \times y)^{-1}$ equals $x^{-1} \times y^{-1}$.
If not, then find a rational number between them

9. For each pair of values in Question 6, are rational numbers $(x \div y)^{-1}$ and $x^{-1} \div y^{-1}$ equal? If not, find a rational number in between.

10. Write true (T) or false (F) for the following statements :

(i) There is no rational number which equals its reciprocal.

(ii) Every rational number has a reciprocal.

(iii) If $|x| = 0$, then x has no reciprocal.

(iv) There is no rational number with absolute value negative.

(v) There is no rational number with reciprocal negative.

(vi) There are infinitely many rational numbers between two rational numbers

(vii) There are infinitely many rational numbers with absolute value less than $\frac{3}{4}$.

(viii) There are infinitely many rational numbers with absolute value less than $\frac{-3}{4}$.

(ix) If $x < y$, then $x^{-1} < y^{-1}$.

(x) If $x < y$, then $|x| < |y|$.

Things to Remember

1. If x and y are rational numbers, then

(i) $x + y$ is a rational number.

(ii) $x - y$ is a rational number.

(iii) $x \times y$ is a rational number.

(iv) $x \div y$ is a rational number, if $y \neq 0$.

(v) $x + y = y + x$

(vi) $x - y \neq y - x$, unless $x = y$.

(vii) $x \times y = y \times x$

(viii) $x \div y \neq y \div x$, unless $x = y \neq 0$.

2. For a rational number x,

(i) $x + 0 = 0 + x = x$

(ii) $x - 0 = x$

(iii) $x \times 0 = 0 \times x = 0$

(iv) $x \times 1 = 1 \times x = x$

(v) $x \div 1 = x$

3. For rational numbers x, y and z,

(i) $(x + y) + z = x + (y + z)$

(ii) $(x \times y) \times z = x \times (y \times z)$

(iii) $(x \div y) \div z \neq x \div (y \div z)$, except $z = 1$

(iv) $x \times (y + z) = x \times y + x \times z$

(v) $x \times (y - z) = x \times y - x \times z$

(vi) $(x + y) \div z = x \div z + y \div z$

(vii) $(x - y) \div z = x \div z - y \div z$

4. (i) If $x = \frac{p}{q}$ is a non-zero rational number, then $x^{-1} = \frac{q}{p}$ is a rational number.

(ii) $(x^{-1})^{-1} = x$

(iii) $x \times x^{-1} = x^{-1} \times x = 1$

5. (i) If x and y are two rational numbers, then $\frac{x + y}{2}$ is a rational number lying between x and y.

(ii) Between x and y, infinitely many rational numbers can be found.

DECIMAL REPRESENTATION OF RATIONAL NUMBERS

CHAPTER 3

3.1 Introduction

In the first Chapter, we introduced rational numbers as a natural extension of the concept of a fraction. We know that a fraction may be represented as a positive decimal number. Likewise, every finite decimal number which is greater than zero may be expressed as a fraction. It is natural if we expect a similar type of relation between rational numbers and decimal numbers. In this Chapter, we explore this relationship. In particular, we show that every rational number may be expressed as a decimal number, and that every decimal number having a finite number of terms in the decimal part may be converted into a rational number. Finally, we shall use decimal representation in computations involving rational numbers.

3.2 Decimal Representation of Rational Numbers

Let us recall that a rational number $\frac{p}{q}$ may be regarded as the number obtained by dividing the integer p by q, $q \neq 0$. If p and q are both positive, i.e., if $\frac{p}{q}$ is a fraction, then we know how to carry out the division of p by q using decimal notation. For example,

$$\frac{3}{4} = 3 \div 4$$

$$\begin{array}{r} 0.75 \\ 4\overline{)3.0} \\ \underline{28} \\ 20 \\ \underline{20} \\ \underline{0} \end{array}$$

Therefore, $\frac{3}{4}$ = .75 or 0.75.

Similarly,

$$\frac{1}{20} = 1 \div 20 \qquad \begin{array}{r} 0.05 \\ 20\overline{)1\ 00} \\ \underline{100} \\ \underline{0} \end{array}$$

Therefore, $\frac{1}{20} = 0.05$

If the rational number is of the form $\frac{-p}{q}$ or $\frac{p}{-q}$, then we find $\frac{p}{q}$ in the decimal notation and put negative sign before it. Thus,

$$\frac{-3}{4} = -0.75,$$

and $$\frac{1}{-20} = -0.05.$$

Example 1: Write the decimal form of

(i) $\frac{17}{5}$ (ii) $\frac{-15}{8}$ (iii) $\frac{19}{-25}$ (iv) $\frac{-23}{-4}$ (v) $\frac{1}{3}$

Solution: (i)

$$\begin{array}{r} 3.4 \\ 5\overline{)17} \\ \underline{15} \\ 20 \\ \underline{20} \\ \underline{0} \end{array}$$

Therefore, $\frac{17}{5}$ = 3.4.

(ii)

$$\begin{array}{r@{}l}
 & 1.875 \\
8 & \overline{)15} \\
 & \underline{8} \\
 & 70 \\
 & \underline{64} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{40} \\
 & \underline{0}
\end{array}$$

Therefore, $\dfrac{15}{8} = 1.875$, and consequently

$$\frac{-15}{8} = -1.875$$

(iii)

$$\begin{array}{r@{}l}
 & 0.76 \\
25 & \overline{)19.0} \\
 & \underline{175} \\
 & 150 \\
 & \underline{150} \\
 & \underline{0}
\end{array}$$

Thus, $\dfrac{19}{-25} = -0.76$

(iv)

$$\begin{array}{r@{}l}
 & 5.75 \\
4 & \overline{)23} \\
 & \underline{20} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{20} \\
 & \underline{0}
\end{array}$$

Therefore, $\frac{23}{4} = 5.75$. However, we know that $\frac{-23}{-4} = \frac{23}{4}$.

Thus, $\frac{-23}{-4} = 5.75$

(v)

$$\begin{array}{r} 0.333\ldots \\ 3\overline{)1.0} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

Therefore, $\frac{1}{3} = 0.333\ldots$

3.3 Non-terminating Decimal Representation

In the decimal representations discussed above, we find that (i), (ii), (iii), and (iv) have a finite number of terms after the decimal point. This is because the division comes to an end after finite steps. However in (v), the division process does not come to an end. As a consequence, the decimal representation of $\frac{1}{3}$ has infinitely many terms.

Consider the division $\frac{5}{3}$.

$$\begin{array}{r} 1.6 \\ 3\overline{)5} \\ \underline{3} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

Observe that the last remainder 2 is the same as the previous remainder. This means that if we continue the division process further, we shall keep on getting the digit 6 repeatedly in the quotient. Hence, $\frac{5}{3} = 1.6666\ldots$.

Consider another example of division. If we have a rational number $\frac{2}{11}$, then we have

$$\begin{array}{r} 0.18 \\ 11\overline{)2.0} \\ \underline{11} \\ 90 \\ \underline{88} \\ 2 \end{array}$$

Since the remainder 2 is the same as the dividend, the sequence of digits 1, 8 in the quotient would repeat.

Consequently, $\frac{2}{11} = 0.18181818\ \ldots$

Example 2: Find the decimal representation of

(i) $\frac{46}{101}$ (ii) $\frac{1}{7}$

Solution: (i)

$$\begin{array}{r} 0.4554 \\ 101\overline{)46.0} \\ \underline{404} \\ 560 \\ \underline{505} \\ 550 \\ \underline{505} \\ 450 \\ \underline{404} \\ 46 \end{array}$$

Since the remainder 46 is the same as the dividend, the sequence of digits 4, 5, 5, 4 in the quotient would repeat.

Therefore, $\frac{46}{101} = 0.455445544554\ \ldots$

(ii)

$$\begin{array}{r@{}l}
 & 0.142857\\
7 & \overline{)1\,000000}\\
 & \underline{7}\\
 & 30\\
 & \underline{28}\\
 & 20\\
 & \underline{14}\\
 & 60\\
 & \underline{56}\\
 & 40\\
 & \underline{35}\\
 & 50\\
 & \underline{49}\\
 & \,1
\end{array}$$

Since the remainder 1 is the same as the dividend, the sequence of digits 1, 4, 2, 8, 5, 7 in the quotient would repeat.

Therefore, $\frac{1}{7} = 0.142857142857\ldots$

In several examples above, we see that the division process does not terminate. We, therefore, call such representation as *non-terminating decimals.* The number with only a finite number of terms in the decimal part are called *finite* or *terminating decimals.*

In the non-terminating decimal representation for $\frac{1}{3}$, we find that the digit 3 goes on repeating. Similarly, in the number $\frac{2}{11}$, the digits 1 and 8 repeat in this order infinitely many times. While writing such decimals, we put a bar over the repeating part. Thus, we write $\frac{1}{3}$ as $0.\bar{3}$. Likewise, we write

$$\frac{2}{11} = 0.\overline{18}\text{, and } \frac{46}{101} = 0.\overline{4554}$$

Using this notation, we may write

$$\frac{1}{7} = 0.\overline{142857}$$

Here, bar sign over 142857 indicates that the digits 1, 4, 2, 8, 5, 7 in this representation repeat, in this order, unendingly

We know that in any division process, the remainder is always less than the divisor. If the remainder at some stage becomes zero, the process will terminate and we find a finite decimal. In non-terminating case, the remainder is never zero and after a few steps, the remainder repeats itself. If the divisor is 7, the (non-zero) remainder has to be one of 1, 2, 3, 4, 5, 6, and thus after at most 6 steps, some remainder would repeat. Hence, *every rational number is either a finite, i.e., terminating decimal or a non-terminating repeating decimal.*

Example 3: Find the decimal representation of

(i) $\frac{17}{45}$ (ii) $\frac{-3}{7}$

Solution: (i)

$$\begin{array}{r} 0.37\ldots \\ 45\overline{)17.0} \\ \underline{135} \\ 350 \\ \underline{315} \\ 35 \end{array}$$

Since the remainder 35 is the same as the previous remainder, the digit 7 would repeat in the quotient.

Therefore, $\frac{17}{45} = 0.3\bar{7}$.

(ii)

$$\begin{array}{r} 0.428571\ldots \\ 7\overline{)3.0} \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \\ 40 \\ \underline{35} \\ 50 \\ \underline{49} \\ 10 \\ \underline{7} \\ 3 \end{array}$$

The remainder 3 is the same as the dividend. Hence, the digits 4, 2, 8, 5, 7, 1 in the quotient would start repeating.

Therefore, $\frac{3}{7} = 0.\overline{428571}$, and consequently

$$\frac{-3}{7} = -0.\overline{428571}$$

Having seen that every rational number can be written as a finite, i.e., a terminating decimal or as a non-terminating repeating decimal, a natural question arises: Which rational numbers can be written as finite, i.e., terminating decimals and which rational numbers have non-terminating repeating decimal representations ? We have seen that representation of rational numbers $\frac{3}{4}, \frac{17}{5}, \frac{1}{20}$ are all terminating decimals. What is so special about these rational numbers ? The denominators 4, 5, 20 are multiples of 2 and /or 5 only. If the denominators of a rational number $\frac{p}{q}$ written in the lowest form has only 2 and/or 5 as factors, then multiplication of q with suitable powers of 2 and 5, will result in a multiple of 10. Thus, $\frac{p}{q}$ is equivalent to a rational number with denominator a power of 10. Thus, $\frac{p}{q}$ can be written as a (finite) decimal number. As illustrations, we have:

$$\frac{3}{4} = \frac{3}{2^2} = \frac{3\times5^2}{2^2\times5^2} = \frac{3\times25}{10^2} = \frac{75}{100} = 0.75$$

$$\frac{17}{5} = \frac{17\times2}{5\times2} = \frac{34}{10} = 3.4$$

$$\frac{1}{20} = \frac{1}{2^2\times5} = \frac{1\times5}{2^2\times5\times5} = \frac{5}{2^2\times5^2} = \frac{5}{100} = 0.05$$

Now, let us consider some numbers which have non-terminating repeating decimals, $\frac{1}{3}, \frac{2}{11}, \frac{46}{101}$ are some such numbers. We observe here that prime factors of their denominators are different from 2 and 5. We know that if q has a prime factor other than 2 and 5, then q cannot be converted, by multiplication, into a power of 10. Consequently, such rational numbers $\frac{p}{q}$ can never be equivalent to a rational number with denominator a power of 10 and, therefore, cannot be represented as a terminating decimal. Representation of such numbers would be non-terminating repeating decimals. Thus, for a rational number $\frac{p}{q}$ in the lowest form, we have

$\frac{p}{q}$ *= finite, i.e., terminating decimal, if prime factors of* q *are 2 or 5 or both, and* $\frac{p}{q}$ *= non-terminating repeating decimal, if* q *has a prime factor which is neither 2 nor 5.*

EXERCISE 3.1

1. Convert the following rational numbers into decimals :

(i) $\frac{15}{4}$ (ii) $\frac{51}{25}$ (iii) $\frac{481}{200}$ (iv) $\frac{321}{40}$

(v) $-\frac{1}{2}$ (vi) $-\frac{13}{20}$ (vii) $\frac{17}{200}$ (viii) $\frac{-21}{16}$

2. Which of the following are terminating decimals and why? Answer without actual division.

(i) $15 \div 17$ (ii) $6 \div 5$ (iii) $7 \div 6$ (iv) $25 \div 0.7$

(v) $9 \div 8$ (vi) $42 \div 9$ (vii) $0.57 \div 0.11$ (viii) $12.5 \div 4$

3. Express as non-terminating repeating decimals :

(i) $\frac{10}{11}$ (ii) $\frac{2}{13}$ (iii) $\frac{17}{90}$ (iv) $\frac{1}{9}$

(v) $\frac{1}{37}$ (vi) $\frac{-20}{3}$ (vii) $\frac{22}{7}$ (viii) $\frac{-28}{21}$

4. Write true (T) or false (F) for the following statements :

(i) $\frac{p}{q}$ has a terminating decimal representation, if q is a prime.

(ii) If $\frac{p}{q}$ and $\frac{r}{s}$ are terminating decimals, then $\frac{p}{q}+\frac{r}{s}$ is also a terminating decimal.

(iii) If $\frac{p}{q}$ and $\frac{r}{s}$ both have non-terminating repeating decimal representations, then so does $\frac{p}{q}+\frac{r}{s}$.

(iv) If $\frac{p}{q}$ and $\frac{r}{s}$ both have terminating decimal representations, then so does $\frac{p}{q}\times\frac{r}{s}$.

(v) If $\frac{p}{q}$ has non-terminating repeating decimal representation and $\frac{r}{s}$ has terminating decimal representation, then $\frac{p}{q}\div\frac{r}{s}$ may have terminating decimal representation.

3.4 Computations with Rational Numbers

We have seen that every rational number can be represented either as a terminating decimal or as a non-terminating repeating decimal We shall now consider the reverse process. Given a terminating decimal number, we can express it as a rational number $\frac{p}{q}$ in its lowest form. For example, the number 1 5 in expanded form is written as

$$1.5 = 1+\frac{5}{10}=\frac{10+5}{10}=\frac{15}{10}=\frac{3}{2}$$

Similarly,

$$5\ 03 = 5+\frac{0}{10}+\frac{3}{100}=\frac{503}{100}$$

$$49.72 = 4\times10+9+\frac{7}{10}+\frac{2}{100}=\frac{4972}{100}$$

$$0.0006 = \frac{6}{10000}, \text{ etc}$$

Example 4: Express the following decimal numbers as rational numbers :

(i) 0.45 (ii) 0.758 (iii) 2.14285 (iv) -3.25

Solution: (i) $0.45 = \frac{4}{10} + \frac{5}{100} = \frac{45}{100} = \frac{9}{20}$

(ii) $0.758 = \frac{7}{10} + \frac{5}{100} + \frac{8}{1000} = \frac{758}{1000} = \frac{379}{500}$

(iii) $2.14285 = \frac{214285}{100000} = \frac{42857}{20000}$

(iv) $-3.25 = \frac{-325}{100} = \frac{-65}{20} = \frac{-13}{4}$

Example 5: Find the sum of the following numbers and express the result in the form $\frac{p}{q}$:

$0.4, -0.37, \quad 0.54, -0.76$

Solution : The required sum is $0.4 + (-0.37) + 0.54 + (-0.76)$

$$= (0.4 + 0.54) + (-0.37 - 0.76)$$
$$= 0.94 + (-1.13)$$
$$= -0.19$$
$$= \frac{-19}{100}$$

Example 6: Find the following sum

$0.39 + 0.750 + 2.15 + (-1.001)$

(i) as a decimal number and convert it into a rational number.

(ii) by converting decimal numbers into rational numbers.

Solution : (i) $0.39 + 0.750 + 2.15 + (-1.001)$

$$= 3.290 - 1.001$$
$$= 2.289$$
$$= \frac{2289}{1000}$$

(ii) $0.39 + 0.750 + 2.15 + (-1.001)$

$$= \frac{39}{100} + \frac{750}{1000} + \frac{215}{100} + \left(-\frac{1001}{1000}\right)$$

$$= \frac{3290}{1000} - \frac{1001}{1000}$$

$$= \frac{3290 - 1001}{1000}$$

$$= \frac{2289}{1000}$$

EXERCISE 3.2

1. Convert the following decimals into rational numbers in the standard form :

(i) 2.8 (ii) 0.037 (iii) -0.75

(iv) -8.625 (v) -0.79 (vi) 7.543

(vii) 9.6 (viii) -5.875

2. Add the following numbers and express the result in the form $\frac{p}{q}$:

(i) 3.8 and 8.7 (ii) -5.4 and 9.8

(iii) -0.82 and 8.6 (iv) 26.86 and -3.92

(v) $3.007, 0.587$ and 18.341 (vi) $3.4, 2.025, 9.36$ and 3.6221

3. Simplify and put the result as $\frac{p}{q}$:

(i) $28.796 - 13.42 - 2.555$

(ii) $36 - 18.59 - 3.2$

(iii) $32.8 - 13 - 10.725 + 3.517$

(iv) $(6.25 + 0.36) - (17.2 - 8.97)$

(v) $879.4 - (87.94 - 8.794)$

4. Evaluate and express the result as $\frac{p}{q}$:

(i)	16.32×28	(ii)	$0.84 \times 2.2 \times 4$
(iii)	$(2.1)^2 \times (1.5)^2$	(iv)	$2.4 \times 3.5 \times 4.8$
(v)	$0.3 \times 0.03 \times 0.003$		

5. Evaluate and express the resulting number as a rational number :

(i)	$4.32 \div 1.2$	(ii)	$4.8432 \div 0.08$
(iii)	$(1.2)^2 \times (0.9)^2 \div 1000$	(iv)	$6.3 \div (0.3)^2$
(v)	$257.894 \div 0.169$		

Things to Remember

1. Every decimal number, having a finite number of terms in the decimal part, is a rational number.
2. Every rational number has a decimal representation.
3. $\frac{p}{q}$ is a terminating decimal, if in its lowest form, the denominator has only 2 and 5 as factors.
4. $\frac{p}{q}$ in the lowest form, has a non-terminating repeating decimal representation, if a prime different from 2 and 5 is a factor of q.
5. A rational number $\frac{p}{q}$ can be written in several equivalent forms, but its decimal representation is unique.

EXPONENTS

CHAPTER 4

4.1 Introduction

In case of integers, we have learnt about exponential notation or power notation. For example,

$$5^4 = 5 \times 5 \times 5 \times 5$$

$$(-3)^3 = (-3) \times (-3) \times (-3)$$

We read 5^4 as 'five raised to the power four' or '4th power of five'. Likewise, $(-3)^3$ is read as '-3 raised to the power 3'. In 5^4, the number 5 is called the *base* and the number 4 is called the *exponent* or simply the *power*. In $(-3)^3$, the base is -3 and the exponent, power or index is 3. In this Chapter, we extend these notations to rational numbers. We also define negative exponents in this Chapter. The exponents are useful in writing very large and very small numbers. We conclude this Chapter by expressing large and small numbers using positive and negative powers of 10 respectively.

4.2 Positive Exponents

Consider a rational number, say $\frac{3}{4}$. The product of this rational number with itself is written in exponential form as $\left(\frac{3}{4}\right)^2$, i.e ,

$$\frac{3}{4} \times \frac{3}{4} = \left(\frac{3}{4}\right)^2$$

Similarly, when $\frac{3}{4}$ is multiplied with itself 5 times, then

$$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \left(\frac{3}{4}\right)^5$$

Likewise,

$$\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)=\left(-\frac{1}{5}\right)^3$$

$$\frac{-43}{79}\times\frac{-43}{79}\times\frac{-43}{79}\times\frac{-43}{79}=\left(\frac{-43}{79}\right)^4, \text{ etc.}$$

In general, if x is a rational number, then m times the product of x with itself is denoted by x^m, i.e., $x \times x \times x \times x \times \ldots$ m times $= x^m$. As earlier, we call x as the *base* and m as the *exponent* or simply the *power* of x

Using the definition of multiplication of rational numbers, we can write

$$\left(\frac{3}{4}\right)^2 = \frac{3}{4}\times\frac{3}{4} = \frac{3\times3}{4\times4} = \frac{3^2}{4^2}$$

$$\left(\frac{3}{4}\right)^5 = \frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4} = \frac{3\times3\times3\times3\times3}{4\times4\times4\times4\times4} = \frac{3^5}{4^5}$$

$$\left(-\frac{1}{5}\right)^3 = \left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right)\times\left(-\frac{1}{5}\right) = \frac{(-1)\times(-1)\times(-1)}{5\times5\times5} = \frac{(-1)^3}{5^3}$$

In general,

$$\left(\frac{p}{q}\right)^m = \frac{p}{q}\times\frac{p}{q}\times \ldots\ m \text{ times}$$

$$= \frac{p\times p\times\ldots\ m\ times}{q\times q\times\ldots\ m times} = \frac{p^m}{q^m}$$

Therefore, for a rational number $\frac{p}{q}$ and for a positive integer m,

$$\boxed{\left(\frac{p}{q}\right)^m = \frac{p^m}{q^m}}$$

It follows from the above relation that

(i) any power of a rational number is a rational number, i.e., $\left(\frac{p}{q}\right)^m$ is a rational number,

(ii) the numerator of the power of a number is the power of the numerator of the number , i.e., numerator of $\left(\frac{p}{q}\right)^m = p^m$, and

(iii) the denominator of the power is the power of the denominator, i.e., denominator of $\left(\frac{p}{q}\right)^m = q^m$.

Using the above *boxed relationship*, we can express a given power of a rational number as a rational number in the standard form and vice-versa, if possible.

Example 1 : Express the following as rational numbers :

(i) $\left(\frac{7}{9}\right)^3$ (ii) $\left(\frac{-5}{11}\right)^4$ (iii) $\left(\frac{69}{72}\right)^2$ (iv) $\left(\frac{21}{-25}\right)^3$

Solution : (i) $\left(\frac{7}{9}\right)^3 = \frac{7^3}{9^3} = \frac{343}{729}$

(ii) $\left(\frac{-5}{11}\right)^4 = \frac{(-5)^4}{11^4} = \frac{625}{14641}$

(iii) $\left(\frac{69}{72}\right)^2 = \frac{69^2}{72^2} = \frac{4761}{5184} = \frac{529}{576}$

(iv) $\left(\frac{21}{-25}\right)^3 = \frac{21^3}{(-25)^3} = \frac{9261}{-15625} = \frac{-9261}{15625}$

Example 2 . Express

(i) $\frac{81}{625}$ as a power of $\left(\frac{3}{5}\right)$.

(ii) $\frac{-49}{64}$ as a power of $\left(\frac{7}{8}\right)$.

(iii) $\frac{-343}{729}$ as a power of $\left(\frac{-7}{9}\right)$.

Solution : (i) $\frac{81}{625}=\frac{3\times3\times3\times3}{5\times5\times5\times5}=\frac{3^4}{5^4}=\left(\frac{3}{5}\right)^4$

(ii) $\frac{-49}{64}=-\frac{7\times7}{8\times8}=-\frac{7^2}{8^2}=-\left(\frac{7}{8}\right)^2$

(iii) $\frac{-343}{729}=-\frac{7\times7\times7}{9\times9\times9}=-\frac{7^3}{9^3}=-\left(\frac{7}{9}\right)^3=\left(\frac{-7}{9}\right)^3$

Example 3 : Simplify the following :

(i) $\left(-\frac{1}{5}\right)^3\times\left(-\frac{1}{5}\right)^2$ (ii) $\left(\frac{-2}{3}\right)^3\times\left(\frac{4}{-5}\right)^2$ (iii) $\left(-\frac{1}{3}\right)^5\div\left(\frac{2}{3}\right)^3$

Solution : (i) $\left(-\frac{1}{5}\right)^3\times\left(-\frac{1}{5}\right)^2$

$$=\frac{(-1)^3}{5^3}\times\frac{(-1)^2}{5^2}$$

$$=\frac{-1}{125}\times\frac{1}{25}$$

$$=\frac{-1}{3125}$$

(ii) $\left(\frac{-2}{3}\right)^3\times\left(\frac{4}{-5}\right)^2$

$$=\frac{(-2)^3}{3^3}\times\frac{4^2}{(-5)^2}$$

$$= \frac{(-2)\times(-2)\times(-2)}{3\times3\times3} \times \frac{4\times4}{(-5)\times(-5)}$$

$$= \frac{-8}{27}\times\frac{16}{25}$$

$$= \frac{-128}{675}$$

(iii) $\left(-\frac{1}{3}\right)^5 \div \left(\frac{2}{3}\right)^3 = \frac{(-1)^5}{3^5} \div \frac{2^3}{3^3}$

$$= \frac{-1}{3\times3\times3\times3\times3} - \frac{2\times2\times2}{3\times3\times3}$$

$$= \frac{-1}{3\times3\times3\times3\times3}\times\frac{3\times3\times3}{2\times2\times2}$$

$$= \frac{-1}{3\times3}\times\frac{1}{2\times2\times2}$$

$$= \frac{-1}{9\times8}$$

$$= -\frac{1}{72}$$

Example 4 : Find the reciprocal of

(i) $\frac{8}{27}$ (ii) $\frac{-125}{216}$ (iii) $-\frac{675}{392}$

and express the same in power notation.

Solution : (i) The reciprocal of $\frac{8}{27} = \frac{27}{8} = \frac{3\times3\times3}{2\times2\times2} = \frac{3^3}{2^3} = \left(\frac{3}{2}\right)^3$

(ii) Reciprocal of $\frac{-125}{216} = \frac{216}{-125}$

$$= -\frac{2\times2\times2\times3\times3\times3}{5\times5\times5}$$

$$= -\frac{2^3\times3^3}{5^3}$$

$$= -\left(\frac{2\times3}{5}\right)^3$$

$$= -\left(\frac{6}{5}\right)^3$$

$$= \left(\frac{-6}{5}\right)^3$$

(iii) The reciprocal of $-\frac{675}{392} = -\frac{392}{675}$

$$= -\frac{2\times2\times2\times7\times7}{3\times3\times3\times5\times5}$$

$$= -\frac{2^3\times7^2}{3^3\times5^2}$$

$$= -\frac{2^3}{3^3}\times\frac{7^2}{5^2}$$

$$= -\left(\frac{2}{3}\right)^3\times\left(\frac{7}{5}\right)^2$$

Example 5 : Find the absolute value of

(i) $\left(\frac{2}{3}\right)^3$ (ii) $\left(-\frac{2}{3}\right)^4$ (iii) $-\left(\frac{7}{8}\right)^2$

Solution : (i) $\left|\left(\frac{2}{3}\right)^3\right| = \left|\frac{2^3}{3^3}\right| = \left|\frac{2\times2\times2}{3\times3\times3}\right| = \left|\frac{8}{27}\right| = \frac{|8|}{|27|} = \frac{8}{27}$

(ii) $$\left|\left(-\frac{2}{3}\right)^4\right| = \left|\frac{(-2)^4}{3^4}\right| = \left|\frac{(-2)\times(-2)\times(-2)\times(-2)}{3\times3\times3\times3}\right| = \left|\frac{16}{81}\right| = \frac{|16|}{|81|} = \frac{16}{81}$$

(iii) $$\left|-\left(\frac{7}{8}\right)^2\right| = \left|-\frac{7^2}{8^2}\right| = \left|-\frac{7\times7}{8\times8}\right| = \left|-\frac{49}{64}\right| = \frac{|-49|}{|64|} = \frac{49}{64}$$

EXERCISE 4.1

1. Express as a rational number $\frac{p}{q}$:

(i) $\left(\frac{3}{7}\right)^2$ (ii) $\left(\frac{3}{4}\right)^5$ (iii) $\left(-\frac{2}{3}\right)^4$ (iv) $\left(-\frac{5}{9}\right)^3$

2. Simplify the following ,

(i) $\left(\frac{3}{5}\right)^4 \times \left(\frac{1}{3}\right)^3$ (ii) $\left(-\frac{2}{3}\right)^4 \times \left(-\frac{3}{4}\right)^3$

(iii) $\left(\frac{1}{3}\right)^4 \div \left(\frac{1}{9}\right)^6$ (iv) $(-2)^5 \div \left(-\frac{1}{3}\right)^3$

3. Express in power notation :

(i) $\frac{1}{243}$ (ii) $\frac{-16}{729}$

(iii) $\frac{-625}{14641}$ (iv) $\frac{-2401}{-256}$

4. Simplify :

(i) $\left(\frac{3}{4}\right)^3 \times \left(\frac{2}{3}\right)^2$ (ii) $\left(-\frac{1}{2}\right)^3 \times 2^3 \times \left(\frac{3}{4}\right)^2$

(iii) $\left[\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^3\right] \times 2^3$ (iv) $(3^2 - 2^2) \div \left(\frac{1}{5}\right)^2$

5. Find the reciprocal of

(i) $(-3)^5$ (ii) $\left(\frac{3}{4}\right)^4$

(iii) $\left(-\frac{1}{5}\right)^8 + \left(\frac{1}{5}\right)^2$ (iv) $\left(\frac{3}{7}\right)^3 \times \left(\frac{7}{3}\right)^5$

6. Find the absolute value of

(i) $\left(-\frac{1}{3}\right)^3$ (ii) $\left(\frac{2}{7}\right)^5$

(iii) $\left(\frac{5}{-3}\right)^4$ (iv) $\left(\frac{-11}{13}\right)^2$

7. Distinguish between the rational numbers $\frac{3^2}{4}$ and $\left(\frac{3}{4}\right)^2$. Which is smaller of the two?

Insert 12 rational numbers between $\frac{3^2}{4}$ and $\left(\frac{3}{4}\right)^2$

4.3 Laws of Exponents (Positive Powers)

Many times, we need to multiply numbers which have the same base. For example,

$$\begin{aligned} 2^5 \times 2^7 &= (2 \times 2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2) \\ &= 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \\ &= 2^{12} \end{aligned}$$

$$\begin{aligned} (-3)^2 \times (-3)^4 &= \{(-3) \times (-3)\} \times \{(-3) \times (-3) \times (-3) \times (-3)\} \\ &= (-3)^6 \end{aligned}$$

$$\begin{aligned} \left(\frac{4}{5}\right)^2 \times \left(\frac{4}{5}\right)^3 &= \left(\frac{4}{5} \times \frac{4}{5}\right) \times \left(\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5}\right) \\ &= \left(\frac{4}{5}\right)^5 \end{aligned}$$

$$\left(\frac{-6}{11}\right)^4 \times \left(\frac{-6}{11}\right)^9 = \left(\frac{-6}{11} \times \frac{-6}{11} \times \ldots 4 \text{ times}\right) \times \left(\frac{-6}{11} \times \frac{-6}{11} \times \ldots 9 \text{ times}\right)$$

$$= \frac{-6}{11} \times \frac{-6}{11} \times \ldots 13 \text{ times}$$

$$= \left(\frac{-6}{11}\right)^{13}$$

In general, if x is a rational number, then

$$x^m \times x^n = (x \times x \times \ldots m \text{ times}) \times (x \times x \times \ldots n \text{ times})$$

$$= (x \times x \times \ldots m + n \text{ times})$$

$$= x^{m+n}$$

Thus, we observe that if two numbers with the same base are multiplied, the product is obtained by adding the exponents. We thus have the following law for exponents

Law I: *If x is a rational number and m and n are positive integers, then*

$$x^m \times x^n = x^{m+n}.$$

This law can be extended to more products of the same base, i.e.,

$$x^m \times x^n \times x^p \times x^q = x^{m+n+p+q}$$

Now observe the following :

$$5^5 \div 5^3 = \frac{5\times5\times5\times5\times5}{5\times5\times5}$$

$$= 5 \times 5 = 5^2$$

Similarly,

$$(-3)^6 \div (-3)^3 = \frac{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)}$$

$$= (-3) \times (-3) \times (-3)$$

$$= (-3)^3$$

$$\left(\frac{4}{7}\right)^5 \div \left(\frac{4}{7}\right)^2 = \frac{\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}}{\frac{4}{7}\times\frac{4}{7}}$$

$$= \frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}$$

$$= \left(\frac{4}{7}\right)^3$$

$$\left(\frac{-2}{3}\right)^4 \div \left(\frac{-2}{3}\right)^2 = \frac{\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}}{\frac{-2}{3}\times\frac{-2}{3}}$$

$$= \frac{-2}{3}\times\frac{-2}{3}$$

$$= \left(\frac{-2}{3}\right)^2$$

The above illustrations suggest the following law :

Law II: *If x is a non-zero rational number, and m and n are positive integers such that $m > n$, then $x^m \div x^n = x^{m-n}$.*

Consider now the following divisions :

$$5^3 \div 5^5 = \frac{5\times5\times5}{5\times5\times5\times5\times5}$$

$$= \frac{1}{5\times5}$$

$$= \frac{1}{5^2}$$

$$(-3)^3 \div (-3)^6 = \frac{(-3)\times(-3)\times(-3)}{(-3)\times(-3)\times(-3)\times(-3)\times(-3)\times(-3)}$$

$$= \frac{1}{(-3)\times(-3)\times(-3)}$$

$$= \frac{1}{(-3)^3}$$

$$\left(\frac{4}{7}\right)^2 \div \left(\frac{4}{7}\right)^5 = \frac{\frac{4}{7}\times\frac{4}{7}}{\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}}$$

$$= \frac{1}{\frac{4}{7}\times\frac{4}{7}\times\frac{4}{7}}$$

$$= \frac{1}{\left(\frac{4}{7}\right)^3}$$

$$\left(\frac{-2}{3}\right)^2 \div \left(\frac{-2}{3}\right)^4 = \frac{\frac{-2}{3}\times\frac{-2}{3}}{\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}}$$

$$= \frac{1}{\frac{-2}{3}\times\frac{-2}{3}}$$

$$= \frac{1}{\left(\frac{-2}{3}\right)^2}$$

The above illustrations suggest the following law .

Law III: *If x is a non-zero rational number and m and n are positive integers such that $m < n$, then* $x^m \div x^n = \frac{1}{x^{n-m}}$.

Let us recall Law I, which says that $x^m \times x^n = x^{m+n}$.

If $n = m$, then

$$x^m \times x^m = x^{m+m}$$

or $$(x^m)^2 = x^{2m}.$$

Similarly,

$$(x^m)^3 = x^m \times x^m \times x^m$$

$$= x^{m+m+m}$$

$$= x^{3m}.$$

$$\begin{aligned}(x^m)^5 &= x^m \times x^m \times x^m \times x^m \times x^m \\ &= x^{m+m+m+m+m} \\ &= x^{5m}\end{aligned}$$

In general, we have

$$(x^m)^n = x^{mn}.$$

Thus, we have the following law :

Law IV: *If x is any non-zero rational number and m and n are positive integers, then* $(x^m)^n = x^{mn}$.

EXERCISE 4.2

1. Fill in the blanks :

(i) $2^3 \times 2^4 = 2^{\cdots}$

(ii) $(-4)^5 \times (-4)^6 = (-4)^{\cdots}$

(iii) $\left(\frac{2}{3}\right)^7 \times \left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^{\cdots}$

(iv) $\left(\frac{3}{4}\right)^8 \div \left(\frac{3}{4}\right)^5 = \left(\frac{3}{4}\right)^{\cdots}$

(v) $(-4)^9 \div (-4)^3 = (-4)^{\cdots}$

(vi) $\left(-\frac{3}{7}\right)^7 \div \left(-\frac{3}{7}\right)^3 = \left(-\frac{3}{7}\right)^{\cdots}$

(vii) $8^{13} \div 8^{19} = \frac{1}{8^{\cdots}}$

(viii) $(-4)^{11} \div (-4)^{15} = \frac{1}{(-4)^{\cdots}}$

2. Simplify :

(i) $\left(\frac{2}{3}\right)^2 \times \left(\frac{2}{3}\right)^3$

(ii) $\left(\frac{-3}{4}\right)^4 \div \left(\frac{-3}{4}\right)^2$

(iii) $(-4)^6 \div (-4)^8$

(iv) $\left(\frac{1}{2^3}\right)^2$

3. Simplify and express the result in power notation :

(i) $\left(\frac{5}{2}\right)^6 \times \left(\frac{5}{2}\right)^2$

(ii) $\left(\frac{11}{3}\right)^{11} \times \left(\frac{11}{3}\right)^2$

(iii) $\left(\frac{3}{4}\right)^7 \div \left(\frac{3}{4}\right)^4$

(iv) $\left(\frac{4}{5}\right)^3 \div \left(\frac{4}{5}\right)^8$

(v) $\left[\left(\frac{2}{3}\right)^4\right]^2$ (vi) $\left[\left(\frac{-3}{4}\right)^3\right]^4$

4. Write true (T) or false (F) for the following statements :

(i) $\left(\frac{-3}{5}\right)^{100} = \frac{-3^{100}}{5^{100}}$ (ii) $\left|\left(\frac{-7}{80}\right)^{80}\right| = \left|\left(\frac{7}{80}\right)^{80}\right|$

(iii) The reciprocal of $\left(\left(\frac{5}{9}\right)^{30}\right)$ is $\left(\frac{9}{5}\right)^{30}$.

(iv) $(10^{10})^{10} = 100^{10}$

(v) $\left(\left(\frac{1}{7}\right)^7\right)^7$ is the reciprocal of 7^{49}.

(vi) $(30 + 30)^{30} = 30^{30} + 30^{30}$

4.4 Laws of Exponents (Negative Powers)

What do we mean by 3^{-2} ? We did not raise this question while dealing with powers of integers. The numbers that we get by raising negative exponents to integers are in fact not integers, but rational numbers. Now with rational numbers available, we may answer this question.

We denoted the reciprocal $\frac{q}{p}$ of a rational number $\frac{p}{q}$ by the symbol $\left(\frac{p}{q}\right)^{-1}$, i.e., the number raised to the exponent -1. There we used 'exponent -1' as a symbol to denote reciprocal. Now, we take the reciprocal of a number as the meaning of 'number raised to -1'. In other words, we say that

$$\boxed{\left(\frac{p}{q}\right)^{-1} = \text{the reciprocal of } \frac{p}{q}}$$

i.e., $$\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$$

Therefore,

$$3^{-1} = \left(\frac{3}{1}\right)^{-1} = \frac{1}{3}$$

$$(-7)^{-1} = \left(\frac{-7}{1}\right)^{-1} = \frac{1}{-7} = -\frac{1}{7}$$

$$x^{-1} = \frac{1}{x}, \ x \text{ being any non-zero integer.}$$

In a similar manner, we say that 3^{-2} is the reciprocal of 3^2, $(-7)^{-4}$ is the reciprocal of $(-7)^4$, x^{-m} is the reciprocal of x^m etc., i.e.,

$$3^{-2} = \frac{1}{3^2}$$

$$(-7)^{-4} = \frac{1}{(-7)^4}$$

$$x^{-m} = \frac{1}{x^m}, \ x \text{ being any non-zero integer.}$$

This meaning can now be applied to rational numbers. Therefore,

$$\left(\frac{7}{9}\right)^{-3} = \text{the reciprocal of } \left(\frac{7}{9}\right)^3$$

$$= \text{the reciprocal of } \frac{7^3}{9^3} = \frac{9^3}{7^3} = \left(\frac{9}{7}\right)^3$$

$$\left(\frac{-15}{23}\right)^{-9} = \text{the reciprocal of } \left(\frac{-15}{23}\right)^9 = \text{the reciprocal of } \frac{(-15)^9}{(23)^9}$$

$$= \frac{23^9}{(-15)^9} = \left(-\frac{23}{15}\right)^9.$$

We thus have the following law :

Law I: *If x is a non-zero rational number and m is a positive integer, then* x^{-m} *is the reciprocal of* x^m, *i.e.,*

$$x^{-m} = \frac{1}{x^m}.$$

Example 6: Find the value of

(i) $\left(\frac{2}{5}\right)^{-6}$ (ii) $\left(\frac{-7}{9}\right)^{-2}$ (iii) $\left(\frac{-3}{11}\right)^{-3}$

Solution : (i) $\left(\frac{2}{5}\right)^{-6} = \frac{1}{\left(\frac{2}{5}\right)^{6}}$

$$= \frac{1}{\frac{2^6}{5^6}}$$

$$= \frac{5^6}{2^6}$$

$$= \frac{5\times5\times5\times5\times5\times5}{2\times2\times2\times2\times2\times2}$$

$$= \frac{15625}{64}$$

(ii) $\left(\frac{-7}{9}\right)^{-2} = \frac{1}{\left(\frac{-7}{9}\right)^{2}} = \frac{1}{\frac{(-7)^2}{9^2}}$

$$= \frac{9^2}{(-7)^2}$$

$$= \frac{81}{49}$$

(iii) $$\left(\frac{-3}{11}\right)^{-3} = \frac{1}{\left(\frac{-3}{11}\right)^{3}}$$

$$= \frac{1}{\frac{(-3)^3}{11^3}}$$

$$= \frac{11^3}{(-3)^3}$$

$$= -\frac{1331}{27}$$

Example 7 : Simplify :

(i) $\left(\frac{4}{3}\right)^{-5} \div \left(\frac{2}{3}\right)^{-3}$ (ii) $\left(\frac{5}{8}\right)^{-7} \times \left(\frac{8}{5}\right)^{-5}$

Solution : (i) $$\left(\frac{4}{3}\right)^{-5} \div \left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{4}\right)^{5} \div \left(\frac{3}{2}\right)^{3}$$

$$= \frac{3\times3\times3\times3\times3}{4\times4\times4\times4\times4} \div \frac{3\times3\times3}{2\times2\times2}$$

$$= \frac{3\times3\times3\times3\times3}{4\times4\times4\times4\times4} \times \frac{2\times2\times2}{3\times3\times3}$$

$$= \frac{3\times3}{2\times4\times4\times4}$$

$$= \frac{9}{128}$$

(ii) $$\left(\frac{5}{8}\right)^{-7} \times \left(\frac{8}{5}\right)^{-5} = \left(\frac{8}{5}\right)^{7} \times \left(\frac{5}{8}\right)^{5}$$

$$= \frac{8\times8\times8\times8\times8\times8\times8}{5\times5\times5\times5\times5\times5\times5}\times\frac{5\times5\times5\times5\times5}{8\times8\times8\times8\times8}$$

$$= \frac{8\times8}{5\times5} = \frac{64}{25}$$

Example 8 : Show that

(i) $3^7 \times 3^{-5} = 3^{7+(-5)}$

(ii) $(-8)^{-9} \times (-8)^5 = (-8)^{-9+5}$

(iii) $\left(\frac{-3}{7}\right)^{-3} \times \left(\frac{-3}{7}\right)^{-4} = \left(\frac{-3}{7}\right)^{-3+(-4)}$

Solution : (i) LHS $= 3^7 \times 3^{-5} = 3^7 \times \frac{1}{3^5} = 3^2$

RHS $= 3^{7+(-5)} = 3^2$

Hence, $3^7 \times 3^{-5} = 3^{7+(-5)}$

(ii) LHS $= (-8)^{-9} \times (-8)^5 = \frac{1}{(-8)^9} \times (-8)^5$

$$= \frac{1}{(-8)^4}$$

$$= (-8)^{-4}$$

RHS $= (-8)^{-9+5} = (-8)^{-4}$

Hence, $(-8)^{-9} \times (-8)^5 = (-8)^{-9+5}$

(iii) LHS $= \left(\frac{-3}{7}\right)^{-3} \times \left(\frac{-3}{7}\right)^{-4} = \left(\frac{-7}{3}\right)^{3} \times \left(\frac{-7}{3}\right)^{4}$

$$= \left(\frac{-7}{3}\right)^{3+4}$$

$$= \left(\frac{-3}{7}\right)^{-7}$$

$$\text{RHS} = \left(\frac{-3}{7}\right)^{-3+(-4)} = \left(\frac{-3}{7}\right)^{-7}$$

Hence, $$\left(\frac{-3}{7}\right)^{-3} \times \left(\frac{-3}{7}\right)^{-4} = \left(\frac{-3}{7}\right)^{-3+(-4)}$$

The above example suggests the following law :

Law II : *If x is a non-zero rational number and m and n are any integers (positive or negative), then*

$$x^m \times x^n = x^{m+n}.$$

Note : If we compare the above law with Law I of Section 4.3, we find that the laws are same except that the Law I of Section 4.3 is stated only for positive integers. In this sense, Law II includes Law I of Section 4.3.

Example 9 : Show that

(i) $$\left(\left(\frac{3}{7}\right)^{-3}\right)^{4} = \left(\frac{3}{7}\right)^{-3\times 4}$$

(ii) $$\left(\left(\frac{8}{11}\right)^{2}\right)^{-5} = \left(\frac{8}{11}\right)^{2\times(-5)}$$

(iii) $$\left(\left(\frac{13}{17}\right)^{-7}\right)^{-8} = \left(\frac{13}{17}\right)^{(-7)\times(-8)}$$

Solution : (i) $$\text{LHS} = \left(\left(\frac{3}{7}\right)^{-3}\right)^{4} = \left(\left(\frac{7}{3}\right)^{3}\right)^{4}$$

$$= \left(\frac{7}{3}\right)^{12}$$

Also, $$\text{RHS} = \left(\frac{3}{7}\right)^{-3\times 4} = \left(\frac{3}{7}\right)^{-12}$$

$$= \left(\frac{7}{3}\right)^{12}$$

Hence, $$\left(\left(\frac{3}{7}\right)^{-3}\right)^{4} = \left(\frac{3}{7}\right)^{-3\times 4}$$

(ii) LHS $= \left(\left(\frac{8}{11}\right)^2\right)^{-5} = \left(\frac{8^2}{11^2}\right)^{-5}$

$$= \left(\frac{11^2}{8^2}\right)^5$$

$$= \frac{11^{2\times 5}}{8^{2\times 5}}$$

$$= \frac{11^{10}}{8^{10}}$$

Also, RHS $= \left(\frac{8}{11}\right)^{2\times(-5)} = \left(\frac{8}{11}\right)^{-10}$

$$= \left(\frac{11}{8}\right)^{10}$$

$$= \frac{11^{10}}{8^{10}}$$

Therefore, $\left(\left(\frac{8}{11}\right)^2\right)^{-5} = \left(\frac{8}{11}\right)^{2\times(-5)}$

(iii) LHS $= \left(\left(\frac{13}{17}\right)^{-7}\right)^{-8} = \left(\left(\frac{17}{13}\right)^7\right)^{-8}$

$$= \left(\frac{17^7}{13^7}\right)^{-8}$$

$$= \left(\frac{13^7}{17^7}\right)^8$$

$$= \frac{13^{7\times 8}}{17^{7\times 8}} = \frac{13^{56}}{17^{56}}$$

Also, $\text{RHS} = \left(\frac{13}{17}\right)^{(-7)\times(-8)} = \left(\frac{13}{17}\right)^{56}$

$$= \frac{13^{56}}{17^{56}}$$

Therefore, $\left(\left(\frac{13}{17}\right)^{-7}\right)^{-8} = \left(\frac{13}{17}\right)^{(-7)\times(-8)}$

The above example suggests the following law :

Law III: *If x is a non-zero rational number and m and n are integers, then* $(x^m)^n = x^{m\times n}$.

Comparing Law IV of Section 4.3 with the above law, we observe that Law IV permits only positive integers, while in the above law, m and n may take negative values also. Thus, the above law is more general to apply.

We now deduce an interesting consequence of the above law. Consider the following :

$$2^3 \times 2^{-3} = 2^{3+(-3)} = 2^0, \text{ as per Law III}$$

Also, $2^3 \times 2^{-3} = 2^3 \times \frac{1}{2^3} = 1$ (By simplification)

i.e., $2^0 = 1$

Likewise,

$$(-3)^5 \times (-3)^{-5} = (-3)^{5+(-5)} = (-3)^0$$

and $(-3)^5 \times (-3)^{-5} = (-3)^5 \times \frac{1}{(-3)^5} = 1$,

i.e., $(-3)^0 = 1$.

The above examples suggest that if x is any non-zero integer, then

$$x^0 = 1.$$

However, if x is a non-zero rational number $\frac{p}{q}$, then

$$x^0 = \left(\frac{p}{q}\right)^0 = \frac{p^0}{q^0} = \frac{1}{1} = 1$$

Therefore, for non-zero rational numbers too, we have $x^0 = 1$ We thus have the following law :

If x is a non-zero rational number, then

$$x^0 = 1.$$

Let us now consider power of a product. If x and y are two integers, then

$$\begin{aligned}(x \times y)^2 &= (x \times y) \times (x \times y)\\ &= (x \times x) \times (y \times y)\\ &= x^2 \times y^2\end{aligned}$$

Similarly,

$$\begin{aligned}(x \times y)^5 &= (x \times y) \times (x \times y) \times (x \times y) \times (x \times y) \times (x \times y)\\ &= (x \times x \times x \times x \times x) \times (y \times y \times y \times y \times y)\\ &= x^5 \times y^5\end{aligned}$$

Thus, for any positive integer m, we have

$$\boxed{(x \times y)^m = x^m \times y^m}$$

Consequently, we have

$$(x \times y)^{-m} = \frac{1}{(x \times y)^m} = \frac{1}{x^m \times y^m} = \frac{1}{x^m} \times \frac{1}{y^m}$$

or $$\boxed{(x \times y)^{-m} = x^{-m} \times y^{-m}}$$

Example 10 : Show that

(i) $$\left(\frac{7}{11} \times \frac{8}{3}\right)^3 = \left(\frac{7}{11}\right)^3 \times \left(\frac{8}{3}\right)^3$$

(ii) $$\left(\frac{-13}{15} \times \frac{19}{16}\right)^{-7} = \left(\frac{-13}{15}\right)^{-7} \times \left(\frac{19}{16}\right)^{-7}$$

Solution : (i) LHS $= \left(\frac{7}{11} \times \frac{8}{3}\right)^3 = \left(\frac{7 \times 8}{11 \times 3}\right)^3$

$$= \frac{(7\times8)^3}{(11\times3)^3}$$

$$= \frac{7^3\times8^3}{11^3\times3^3}$$

Also, $$\text{RHS} = \left(\frac{7}{11}\right)^3 \times \left(\frac{8}{3}\right)^3 = \frac{7^3}{11^3}\times\frac{8^3}{3^3}$$

$$= \frac{7^3\times8^3}{11^3\times3^3}$$

Thus, $\left(\frac{7}{11}\times\frac{8}{3}\right)^3 = \left(\frac{7}{11}\right)^3 \times \left(\frac{8}{3}\right)^3$

(ii) $$\text{LHS} = \left(\frac{-13}{15}\times\frac{19}{16}\right)^{-7} = \left(\frac{-13\times19}{15\times16}\right)^{-7}$$

$$= \left(\frac{15\times16}{-13\times19}\right)^7$$

$$= \frac{(15\times16)^7}{(-13\times19)^7}$$

$$= \frac{15^7\times16^7}{(-13)^7\times(19)^7}$$

$$\text{RHS} = \left(\frac{-13}{15}\right)^{-7} \times \left(\frac{19}{16}\right)^{-7} = \left(\frac{15}{-13}\right)^7 \times \left(\frac{16}{19}\right)^7$$

$$= \frac{15^7}{(-13)^7}\times\frac{16^7}{19^7}$$

$$= \frac{15^7\times16^7}{(-13)^7\times19^7}$$

Therefore, $\left(\frac{-13}{15}\times\frac{19}{16}\right)^{-7} = \left(\frac{-13}{15}\right)^{-7}\times\left(\frac{19}{16}\right)^{-7}$

This example suggests the following law :

Law V: *If x and y are non-zero rational numbers and m is any integer, then*

$$(x \times y)^m = x^m \times y^m.$$

Example 11: Find x such that

(i) $\left(\frac{7}{4}\right)^{-3}\times\left(\frac{7}{4}\right)^{-5}=\left(\frac{7}{4}\right)^{x-2}$ (ii) $\left(\frac{125}{8}\right)^{5}\times\left(\frac{125}{8}\right)^{x}=\left(\frac{5}{2}\right)^{18}$.

Solution : (i) $\left(\frac{7}{4}\right)^{-3}\times\left(\frac{7}{4}\right)^{-5}=\left(\frac{7}{4}\right)^{x-2}$

or $\left(\frac{7}{4}\right)^{-3+(-5)}=\left(\frac{7}{4}\right)^{x-2}$ (Law II)

From here, we get

$-3+(-5)=x-2$

i.e., $-8=x-2$

or $-8+2=x-2+2$

or $-6=x$

Thus, $x=-6$

(ii) LHS $=\left(\frac{125}{8}\right)^{5}\times\left(\frac{125}{8}\right)^{x}=\left(\frac{125}{8}\right)^{5+x}$

Also, $\frac{125}{8}=\frac{5\times5\times5}{2\times2\times2}=\frac{5^3}{2^3}=\left(\frac{5}{2}\right)^{3}$

Therefore, $\left(\frac{125}{8}\right)^{5+x}=\left(\left(\frac{5}{2}\right)^{3}\right)^{5+x}=\left(\frac{5}{2}\right)^{3\times(5+x)}=\left(\frac{5}{2}\right)^{15+3x}$

Equating with RHS, we get

$$\left(\frac{5}{2}\right)^{15+3x}=\left(\frac{5}{2}\right)^{18}$$

$$\therefore \quad 15 + 3x = 18$$

$$\text{or} \quad 3x = 3$$

$$\text{or} \quad x = 1$$

EXERCISE 4.3

1. Find the value of

(i) 2^{-3} (ii) $(-3)^{-4}$ (iii) $\left(\frac{1}{4}\right)^{-4}$

(iv) $\left(\frac{2}{3}\right)^{-5}$ (v) $\left(\frac{-3}{5}\right)^{-2}$ (vi) $\left(\frac{4}{-7}\right)^{-3}$

(vii) $\left(\frac{-3}{4}\right)^{-4}$

2. Using the laws of exponents, express each of the following as a rational number with positive exponent :

(i) $\left(\frac{2}{3}\right)^{-5}$ (ii) $(2^{-4})^{2}$ (iii) $4^{3} \times 4^{-5}$

(iv) $\left[\left(\frac{3}{2}\right)^{-2}\right]^{3}$ (v) $2^{-3} \times (-7)^{-3}$ (vi) $(2^{5} \div 2^{8}) \times 2^{-7}$

3. Express as a rational number with negative exponent :

(i) $\left(\frac{1}{2}\right)^{3}$ (ii) $(2^{4})^{3}$ (iii) $4^{3} \times 4^{4}$

(iv) $\left[\left(\frac{-6}{5}\right)^{3}\right]^{2}$ (v) $(-3)^{4} \times \left(\frac{5}{3}\right)^{4}$ (vi) $(3^{7} \div 3^{3}) \times 3^{3}$

4. Find the value of

(i) 3^{0} (ii) 4^{5-5} (iii) $\left(\frac{5}{7}\right)^{4+2-6}$

(iv) $(-3)^{3\times5-6-9}$ (v) $2^0 + 3^0 + 4^0$ (vi) $2^0 \times 3^0 \times 4^0$

(vii) $(3^0 - 2^0) \times 5^0$ (viii) $(6^0 - 2^0) \times (6^0 + 2^0)$

5. By what number should we multiply 3^{-7} so that the product may be equal to 3 ?

6. By what number should $(-4)^5$ be divided so that the quotient may be equal to 4^{-2} ?

7. Find x so that $\left(\frac{5}{3}\right)^{-5} \times \left(\frac{5}{3}\right)^{-11} = \left(\frac{5}{3}\right)^{8x}$.

8. Find m so that $\left(\frac{2}{9}\right)^{3} \times \left(\frac{2}{9}\right)^{-6} = \left(\frac{2}{9}\right)^{2m-1}$.

9. Find the reciprocal of the rational number $\left(\frac{1}{2}\right)^{-2} \div \left(\frac{2}{3}\right)^{-3}$.

10. If $\frac{p}{q} = \left(\frac{3}{2}\right)^{-2} \div \left(\frac{6}{7}\right)^{0}$, find the value of $\left(\frac{p}{q}\right)^{-3}$.

11. Simplify $\left[\left(\frac{2}{3}\right)^{2}\right]^{3} \times \left(\frac{1}{3}\right)^{-2} \times 3^{-1} \times \frac{1}{6}$.

12. Show that

(i) $\left(\frac{9}{13} \times \frac{-11}{17}\right)^{-8} = \left(\frac{13}{9}\right)^{8} \times \left(\frac{17}{-11}\right)^{8}$

(ii) $\left(\frac{-12}{19} \times \frac{-27}{43}\right)^{-7} = \left(\frac{19}{12}\right)^{7} \times \left(\frac{43}{27}\right)^{7}$

13. Write true (T) or false (F) for the following statements :

(i) $a^b > b^a$ is true, if $a = 3$ and $b = 4$; but false, if $a = 2$ and $b = 3$.

(ii) For all integers a and b,

$$(a + b)^2 \geq a^2 + b^2, \text{ and}$$
$$(a - b)^2 \leq a^2 + b^2.$$

(iii) For all non-zero rational numbers x,

reciprocal of x^m = (reciprocal of x)^m.

(iv) The relation $x^m \div x^n = x^{m-n}$ is true if $x > 0$ and $m \geq n$

(v) $x^0 \times x^0 = \frac{x^0}{x^0}$ is true for all non-zero values of x.

4.5 Use of Exponents in Expressing Large and Small Numbers

In many situations, we come across numbers which are very large. For example, the age of universe in years, the mass of Earth in tons, the distance of Sun from Earth (in km) are numbers which are very large Such large numbers are normally approximate, and not exact numbers. Therefore, these may be written as a certain number followed by a number of zeros. For example, the speed of light in vacuum is 299792.5 km per second. It is approximated as 300000 km per second or as 300,000,000 m/s. Similary, the age of our universe is approximated to be 8,000,000,000 years. The mean distance from Earth to Sun is 150,000,000 km. The mass of Earth is 5980,000,000,000,000,000, 000 metric tons.

Such large numbers are usually written using exponents with base 10. For example, the number 300,000,000 may be written as 3×10^8 or as 30×10^7 or 300×10^6. Thus, every large number can be expressed as $k \times 10^n$, where k is some number and n is some *positive* integer. To write 300,000,000 in this form, we may take $k = 3$ and $n = 8$ or $k = 30$ and $n = 7$ or $k = 300$ and $n = 6$, etc. However, for the sake of uniformity, we write the number in the form where k is a terminating decimal number such that $1 \leq k < 10$ and adjust n accordingly. Therefore, using this notation we write the speed of light as 3×10^8 m/s, the age of the universe as 8×10^9 years, the distance of Earth from Sun as 1.5×10^8 km and the mass of the Earth as 5.98×10^{21} metric tons or as 5.98×10^{24} kg, etc.

As exponential notation helps us in writing large numbers, very small numbers are also written using exponential notation, but this time the exponent n of 10 is *negative*. For example, the unit to measure wavelength is *angstrom* and one angstrom is $\frac{1}{10,000,000,000}\,m$ We may express this number as 1×10^{-10} m. Likewise, the number 0.000000000387 can be written as 3.87×10^{-10} or 38.7×10^{-11} or 387×10^{-12}, etc.

Example 12 : Express the following numbers in the form $k \times 10^n$, where k is a number and n is an integer:

(i) 3186500000 (ii) 0.0000837

Solution : We can write

(i) 3186500000 as 31865×10^5

or as 3186.5×10^6

or as 318.65×10^7

or as 31.865×10^8

or as 3.1865×10^9

or as 0.31865×10^{10} and so on

(ii) 0.0000837

$$= \frac{837}{10000000}$$

$$= 837 \times 10^{-7} \text{ or } 83.7 \times 10^{-6} \text{ or } 8.37 \times 10^{-5} \text{ etc.}$$

EXERCISE 4.4

1. Express the following numbers in the form $k \times 10^n$ with the given value of n.

(i) 980000000, $n = 8$

(ii) 0.000000000097, $n = -11$

(iii) 0 00000000000055, $n = -14$

(iv) 10700000000, $n = 9$

(v) 602000000000000000000000, $n = 22, 23$

2. Write the following numbers in the usual form :

(i) 6.5×10^{-6}

(ii) 8.9×10^{-9}

(iii) $5\ 6146929 \times 10^7$

(iv) 5.8×10^{12}

(v) 1.001×10^9

3. Express the numbers appearing in the following statements in the form $k \times 10^n$, where $1 \leq k < 10$ and n is an integer :

(i) Every day, about 1050000 kg of pollutants are emitted in the Capital of India.

(ii) The Earth has about 1,353,000,000 cubic km of sea water and this sea water contains around 1,361,000,000 kg of gold.

(iii) In March 2001, the population of India was approximated to be 1,027,000,000 in which 531,200,000 were males and 495,800,000 females.

(iv) 1 micron is equal to $\frac{1}{1000000}$ m.

Things to Remember

1. If $x = \frac{p}{q}$ is a rational number and m is a positive integer, then
$$x^m = \frac{p^m}{q^m}.$$
2. If m is a positive integer, and $x \neq 0$, then $x^{-m} = (x^{-1})^m$, where x^{-1} is the reciprocal of x.
3. For any non-zero rational number x,
$$x^0 = 1.$$
4. For any integers m and n and any rational number x, $x \neq 0$,
$$x^m \times x^n = x^{m+n}.$$
5. For any integers m and n and for any non-zero rational number x,
$$x^m \div x^n = x^{m-n}.$$
6. For any integers m and n and for any rational number $x \neq 0$,
$$(x^m)^n = x^{m \times n}.$$
7. If x and y are non-zero rational numbers and m is any integer, then
$$(x \times y)^m = x^m \times y^m.$$
8. Every number, large or small, may be expressed in the form $k \times 10^n$, where k is a terminating decimal and n is an integer (positive for large numbers and negative for small numbers). Usually, the value of k is taken such that $1 \leq k < 10$

As History Tells Us

You have learnt in Class VI that all the old civilizations had some kind of numeration system. The oldest civilization, viz. the Sumerian-Babylonian civilization (5000 BC approximately) had only two symbols – '𒁹' for *one* and '𒌋' for *ten* With these symbols and a *gap* (for absence of any symbol), they could write quite big numbers. Upto 60, their system resembled the decimal system. Beyond 60, they used a system something like base 60.

Egyptians civilization that started around 5000 BC also had a set of numerals. This consisted of as many vertical lines for each numeral from 1 to 9 (I, II, III, . . .). For 10, 20, . . 100, 1000, . . . they had separate *symbols*; they never used *script* for numbers, a feature unique to them.

India has the unique distinction of combining the three concepts — *decimal system, place value*, and a *computational zero* (शून्य). It is in the works *Brahm - sphut - siddhanta* (ब्रह्म स्फुट सिद्धान्त) of the Indian mathematician *Brahmagupta* (598 - 678 AD) that operations on zero are described, giving it the status of a number like any other number. Wrote *Brahmagupta*:

" *Sunyayoh khadanayoh khasunya khasunyayorva badha sunyam*"
("शून्ययो खडनयोः खशून्य खशून्ययोर्वा बद्ध शून्यम्")

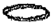
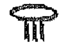
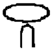
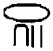
In our notation, this means

$$a \times 0 = 0, \quad 0 \times 0 = 0 \quad \text{and} -a \times 0 = 0$$

That, the invention of this computational zero gave an impetus unmatched to all branches of [illegible] old story so well - known. *Gauss*, one of the five greatest mathematicians of all times wrote :

"It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a position value as well as an absolute value, a pro-and important idea which appears so simple to us now that we ignore its true merit ... we shall appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity."

The story of Indian glory in the field of numbers does not end here. Indian contribution to fractions (and thus indirectly to rational numbers) is to be recognized with great respect. The ancient Indian mathematicians made use of fractions, although, in their symbols, they did not use a fraction bar. *Brahmagupta* gave rules for working with fractions. These rules are quite like the rules we use today From India, the fractions and rules for working with them spread into the Arab world. It was translated into Arabic by the famous Arab mathematician *Al Khowarizmi*. From the Arab world, the Indian work spread to Italy and the West

To the ancient man, the need for rational numbers (or what they called fractions) arose much later than the need for counting numbers and integers. Essentially, the first notion of a fraction was developed in the process of measuring lengths, areas, etc. Babylonians made extensive use of *sexagesimal fractions* (fractions with denominator 60), since their measures and money were divided in terms of 60 parts. Egyptians, it appears, had symbols for unit fractions such as, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, etc. They wrote the symbol  over the number to represent a fraction. Thus,  would represent $\frac{1}{3}$,  would represent $\frac{1}{10}$,  would represent $\frac{1}{12}$, etc

The only other fraction for which they had a symbol was $\frac{2}{3}$. However, they were aware of still other fractions. In *Rhind Papyrus*, we find an evidence of Egyptians using fractions, around 1700 BC; which we would write today as $\frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}$, etc.

Greeks also knew about fractions — they wrote the denominator above the numerator although they did use other symbols for fractions as well. The Romans employed *duodecimal fractions* since their weights and money were divided in terms of 12 parts.

The abstract mathematical concept of negative numbers, alongwith zero, is the most radical invention of mathematics. To begin with it had no physical reality like other type of numbers. It was again *Brahmagupta*, who after the invention of a computational zero, realized that just as positive numbers start with zero and extend to infinity so the negative numbers start with zero and extend to infinity in the other direction.

The lack of physical reality made the acceptance of negative numbers a slow process. At home, even the later mathematicians like *Mahavira* (850 AD) and *Bhaskara* (born 1114 AD) accepted negative numbers hesitantly. But the Arab mathematicians like *Al Khowarizmi* and *Omar Khayyam* accepted the negative numbers in good faith, and spread them.

Once the fractions and negative numbers were there, the conceptualization of rational numbers was not a difficult task. The current presentation and usage of rational numbers has been developed in the previous century.

DIRECT AND INVERSE VARIATIONS

CHAPTER 5

5.1 Introduction

In earlier classes, you have read about *Ratio and Proportion* and *Unitary Method*. In this Chapter, we shall use the concept of proportion and unitary method in the development of the concept of variation. You know that .

1. As the speed of a car increases, the time taken to cover the same distance decreases. (An increase in the speed causes a decrease in the time.)
2. The more the money deposited in a bank, the more is the interest earned. (Money and interest increase together.)
3. As the number of labourers increases, the number of days taken to complete the same work decreases. (Increase in the number of labourers decreases the number of days.)

Thus, we see that sometimes two related quantities increase or decrease together, but sometimes as one quantity increases, the other decreases. This gives rise to the concepts of *direct variation* and *inverse variation*. We introduce these concepts in this Chapter and use the same in solving problems on time and work and on time and distance.

5.2 Direct Variation

Suppose that potatoes cost Rs 10 per kg. To buy two kilograms of these, you have to spend double the amount but to buy half a kilogram, you spend only half as much. The more you buy, the more you spend and the less you buy, the less you spend, but the ratio of the quantity of potatoes to its cost remains the same. Two quantities varying in this manner are said to *vary directly* as each other.

If 5 kg of potatoes cost Rs 50.00, then what would be the cost of 20 kg of potatoes? Your answer certainly is Rs 200. The question is : how did you arrive at it? Using unitary method, you could have first found the cost of one kg of potatoes and then obtained the cost of twenty kg of potatoes. However, you might have noted that since 5 kg of potatoes cost Rs 50, 20 kg of potatoes would cost four times as much. How much would 40 kg of potatoes cost? Well ! 8 times as much. Let us denote the weight of potatoes in kg by x and the corresponding costs in rupees by y. The table below gives some values of x and the corresponding values of y :

Weight of potatoes in kg (x)	5	10	20	30	40	50	60
Cost in rupees (y)	50	100	200	300	400	500	600

Examine the table. As the values of x increase, what happens to the corresponding values of y? Do they also increase? Yes, they do indeed! Also, compare the ratios $\frac{x}{y}$ for various values of x and the corresponding values of y. For example, $\frac{5}{50}, \frac{10}{100}, \frac{20}{200}, \frac{30}{300}$ and so on. What do you observe? Do you always get the same value? Yes you do! Every time, you get the same value $\frac{1}{10}$. In other words, the ratio $\frac{x}{y}$ does not change; it *remains constant*. Note that x and y are positive and hence $\frac{x}{y}$ is always positive.

Let us denote the constant value $\frac{x}{y}$ by k. Then,

$$\frac{x}{y} = k \text{ or } x = ky \qquad (1)$$

Note that k does not change. Therefore, the above relation shows that x and y increase (decrease) together. If we increase the value of either variable (x or y), then we must suitably increase the value of the other variable also in order to balance the two sides of relation (1). If we decrease the value of either variable (x or y), then we must suitably decrease the value of the other variable also in order to balance the two sides of relation (1). For example, doubling x requires doubling y, replacing x by $3x$ requires replacing y by $3y$ and so on Similarly, decreasing the value of x by one - fourth of x means decreasing, the value of y by one-fourth of y. In such cases, we say that x and y vary directly as each other.

Thus, we see that two quantities x and y are said to *vary directly* or are said to be *in direct proportion*, if

1. An increase in x is followed by a corresponding increase in y in such a manner that the ratio $\frac{x}{y}$ remains constant and positive.
2. A decrease in x is followed by a corresponding decrease in y in such a manner that the ratio $\frac{x}{y}$ remains constant and positive.

Let us consider some examples where one quantity varies directly as another.

Example 1 : Find a and b in the following table, if x and y vary directly :

x	10	8	a
y	15	b	24

Solution : It is given that x and y vary directly. Therefore, the ratio $\frac{x}{y}$ should be constant and positive. Now $\frac{10}{15}$, $\frac{8}{b}$, $\frac{a}{24}$ should have the same value. Thus,

$$\frac{10}{15}=\frac{8}{b}=\frac{a}{24}$$

or $$b = 12 \text{ and } a = 16$$

Example 2 : The cost of 6 metres of cloth of a certain type is Rs 240.00. Write in a table the costs of 2, 3, 12 and 18 metres of cloth of the same type. Do the length and cost vary directly as each other?

Solution : We can use the unitary method and get the costs of 2, 3, 12 and 18 metres as 80, 120, 480 and 720 rupees respectively.

We can use our common sense and note that the desired costs are nothing but one-third, one-half, double and triple, respectively of the cost of 6 metres of cloth Thus, the desired costs (in rupees) are $\frac{1}{3}\times 240, \frac{1}{2}\times 240, 2\times 240$ and 3×240 respectively. Whatever way we proceed, we find that the required table is.

l = *length* (*in metres*)	2	3	6	12	18
c = *cost* (*in rupees*)	80	120	240	480	720

Do the two quantities l (length) and c (cost) vary directly? Yes, they do, because

1. As l increases (decreases), so does c.
2. If we form the ratio $\frac{c}{l}$ for corresponding values of c and l, we get $\frac{80}{2}$, $\frac{120}{3}, \frac{240}{6}, \frac{480}{12}, \frac{720}{18}$, and all of these have the same value 40. Thus,

$$\frac{c}{l} = 40 \text{ or } c = 40l \text{ or cost} = 40 \text{ (length)}.$$

Because of the relation *cost* = 40 *times length*, we say that the *cost varies directly as* 40 *times the length*. (Does length vary directly as $\frac{1}{40}$ times the cost?)

Remarks : 1. The relation $c = 40\,l$ ensures that the cost and the length increase (decrease) together. Strictly speaking, in order to examine whether or not c and l vary directly, all we needed was a relation of the type $c = k\,l$ between c and l for a positive k. There was no need to find out whether or not c and l are increasing together. But first examining whether or not they increase (decrease) together is very easy and helpful. However, it should be noted that increasing try to obtain the relation $x = ky$. Otherwise, we need not bother; it is not a case of direct variation as shown in solved example 4.

2. Notice that when two quantities vary directly, then the ratio of two values of one quantity is the same as the ratio of the corresponding values of the other quantity. Thus, in the above example, $\frac{2}{3}=\frac{80}{120}, \frac{2}{18}=\frac{80}{720}$, etc.

Example 3: Seven dozen oranges cost Rs 91.00. Find the cost of 10 dozen oranges.

Solution : This is obviously a case of direct variation. As the number of oranges bought increases, the amount spent also increases in the same ratio. Suppose 10 dozen oranges cost Rs x. Let us write the information as follows :

Number of oranges (in dozens)	7	10
Amount spent (in rupees)	91	x

Since the *number of oranges* and *amount spent* vary directly as each other,

$$\therefore \qquad \frac{7}{91}=\frac{10}{x}$$

$$\text{or} \qquad x=\frac{910}{7}$$

$$\text{or} \qquad x = 130$$

i.e., the cost of 10 dozen oranges is Rs 130.

Example 4: A stone is falling freely from the top of a tower The distance (h) covered and the time (t) taken to fall is given by the formula

$$h=\frac{1}{2}gt^2, \qquad (1)$$

where g is a constant known as acceleration due to gravity. Find

(i) h when $t = 2$ and $t = 4$.

(ii) $\frac{h}{t}$ for $t = 2$ and $t = 4$.

(iii) Do h and t vary directly as each other?

Solution : (i) We have

$$h=\frac{1}{2}gt^2$$

When $t = 2,\ h=\frac{1}{2}g\,.2^2 = 2g$

and when $t = 4,\ h = \frac{1}{2}\ g\ .\ 4^2 = 8g.$

(ii) From (1), we have $\frac{h}{t} = \frac{1}{2}$ gt.

$\therefore$ For $t = 2,\ \frac{h}{t} = \frac{1}{2}g\times 2 = g.$ For $t = 4,\ \frac{h}{t}=\frac{1}{2}g\times 4 = 2g.$

(iii) Note that as t increases, h increases, but the ratio $\frac{h}{t}$ is not constant, and hence h and t do not vary directly as each other.

Remark : In this example, h and t^2 vary directly as each other.

Example 5 : If the weight of 6 sheets of paper be 45 grams, how many sheets of the same paper would weigh $1\frac{1}{2}$ kilograms?

Solution : Our common sense tells us that the more the number of sheets, the more would they weigh. Moreover, the increase in the number and weight would be proportionate. (That is, the two would change in the same ratio.) Hence, these two quantities vary directly as each other and are, therefore, related as

(number of sheets) = k × (weight of sheets in grams),

where k is some positive number. Let us find k. We know that 6 sheets weigh 45 grams. So we must have

$$6 = k \times 45$$

or

$$k=\frac{2}{15}$$

Hence,

$$(\text{number of sheets}) = \frac{2}{15} \times (\text{weight of sheets in grams})$$

Now suppose x sheets weigh $1\frac{1}{2}$ kg, which is 1500 in grams. So,

$$x = \frac{2}{15} \times 1500 = 200$$

Thus, 200 sheets weigh $1\frac{1}{2}$ kg.

Alternatively, we can solve the problem as follows : We form a table as shown below:

Number of sheets	6	x
Weight (in grams)	45	1500

Note that $1\frac{1}{2}$ kg = 1500 g. Thus,

$$\frac{6}{45} = \frac{x}{1500}$$

or $$x = 1500 \times \frac{6}{45} = 200$$

Example 6: A car travels 432 km on 48 litres of petrol How far would it travel on 20 litres of petrol?

Solution : We note first that the lesser the petrol consumed, the smaller the number of kilometres travelled. Moreover, the change is proportionate. Suppose that the car travels x km on 20 litres of petrol Let us write our information in the form of a table as shown below :

Petrol (in l)	48	20
Distance (in km)	432	x

Since the ratio between the amounts of petrol and kilometres covered remains constant, we have

$$48 : 20 = 432 : x$$

The above relation gives the value of x as $\dfrac{20 \times 432}{48}$ or 180

Thus, the car would travel 180 km on 20 litres of petrol.

Example 7 : An electric pole, 24 metres high, casts a shadow of 20 metres. Find the height of a tree, if it casts a shadow of 15 metres under similar conditions.

Solution : It is clear that this is a case of direct variation. Let us suppose that the height of the tree is x metres.

$\therefore$ $$24 : x = 20 : 15$$

or $$24 \times 15 = 20 \times x$$

or $$20x = 360$$

or $$x = 18$$

i.e., the height of the tree is 18 metres.

Example 8 : A train covers 95 kilometres in $1\frac{1}{4}$ hours. Assuming that the speed of the train remains uniform, find the time required to cover a distance of 266 km.

Solution : Our common sense tells us that since the speed remains uniform, therefore the distance covered would increase proportionately with time. Hence, these two quantities vary directly as each other and are, therefore, related as

$$\text{Distance covered} = k\ (\text{Time taken}),$$

where k is some positive number. We know that 95 km are covered in 75 minutes. So we must have

$$95 = k \times 75$$

or $$k = \frac{95}{75}$$

Hence, $$266 = \frac{95}{75} \times (\text{Time taken})$$

or $$\text{Time taken} = \frac{266 \times 75}{95}$$

$$= 210$$

Thus 210 minutes, i.e., 3 hours 30 minutes will be the time required to cover 266 km.

EXERCISE 5.1

1. Replace each star in the following tables by a suitable number, if x and y vary directly:

(i)

x	7	9	13	*	25
y	21	*	*	63	*

(ii)

x	2.5	5	7.5	10	*	12
y	*	5	*	*	11	*

(iii)

x	0.1	0.5	2	*	32
y	*	*	8	32	*

2. The height to which a balloon with hydrogen gas rises in the air varies directly as time. Given below are some observations about the time and the corresponding height of the balloon (in metres). Find the missing terms in the Table.

Time (*in minutes*)	3	4		25	
Height of the balloon (*in metres*)		48	84		1860

3. 15 stamps of equal value cost Rs 18.00. How many stamps of the same value can be bought for Rs 36.00?

4. A machine takes 5 hours in cutting 120 tools. How many tools will it cut in 20 hours?

5. An agent receives a commission of Rs 73.00 on sales of Rs 1000.00. How much commission will he get on sales of Rs 100.00?

6. At a party, 8 bottles of soft drink are served for every batch of 5 children. How many bottles would be served, if 40 children are present at the party?
7. If the thickness of 500 sheets of paper is 3.5 cm, then what would be the thickness of 275 sheets of this paper?
8. If the cost of 93 m of a certain kind of plastic sheet is Rs 1395, then what would it cost to buy 105 m of such plastic sheet?
9. Salma types 540 words during half an hour. How many words would she type in 6 minutes?

5.3 Inverse Variation

We have seen that, in case of direct variation, the two quantities increase or decrease together in the same ratio. Sometimes, corresponding to an increase in the first quantity, there is a decrease in the second, and when there is a decrease in the first quantity, there is a corresponding increase in the second. For example, if 4 persons can do a job in 6 days, then if we put only one person on the same job, he would require four times as much time. In other words, one person would do this job in 24 days. The number of days required for doing the job when 1, 2, 4, 6, 8 or 12 persons are put on it are listed in the following table :

Number of persons	1	2	4	6	8	12
Days required	24	12	6	4	3	2

Thus, as the number of persons increases, the number of days decreases. Put differently, as the number of persons decreases, the number of days increases

We have noted that in case of direct variation, the ratio between any two entries in one row (i.e., two values of one quantity) is exactly the same as that between the corresponding entries (values of the other quantity) in the other row. Let us see what happens here. Take the ratio 2 : 4 between the 2nd and the 3rd values of the upper row in the above table. The corresponding entries 12 and 6 in the lower row are in the ratio 4 : 2. The ratios between the 2nd and the last entries in the two rows are 2 : 12 and 12 : 2. What do you observe? Try with other pairs. Do you find that every time the ratios are the inverses of each other?

Notice that the product of the corresponding values of the two quantities is constant. For example, in the above table, $1 \times 24 = 2 \times 12 = 4 \times 6 = \ldots = k$ say.

Two quantities x and y are said to *vary inversely as each other,* if there exists a relation of the type $xy = k$ between them, k being a fixed positive number.

We also say that *x varies inversely as y and that y varies inversely as x* Note that if $xy = k$, and if y_1, y_2 are the values of y corresponding to the values x_1, x_2 of x respectively, then $x_1 y_1 = x_2 y_2$ (= k) so that $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

Recall that in case of direct variation, we had a relation of the type $\frac{x}{y} = k$.

Example 9: In a hostel of 50 girls, there are food provisions for 40 days. If 30 more girls join the hostel, how long will these provisions last?

Solution: Note that more the girls, the sooner would the provisions exhaust. It is, therefore, the case of inverse variation. The number of girls in the two situations is 50 and (50 + 30) = 80 respectively. If the provisions last for x days when the number of girls increases from 50 to 80, we have the following table:

Number of girls	50	80
Number of days	40	x

Since the variation is inverse, we have

$$50 \times 40 = 80 \times x$$

This gives the value of x as 25.

Thus, the provisions will last for 25 days.

Example 10 : Temperature remaining constant, the volume of the gas varies inversely as pressure. If the volume of the gas at pressure 270 mm is 840 cu cm, find the volume of the gas at pressure 315 mm, assuming that the temperature remains constant throughout the experiment.

Solution : Here, volume and pressure vary inversely. Let the required volume be x cu cm. Then,

$$270 \times 840 = 315 \times x$$

or $$x = \frac{270 \times 840}{315} = 720$$

i.e., the volume of the gas at pressure 315 mm is 720 cu cm.

5.4 Time and Work, Time and Distance

We have already seen that the concepts of direct and inverse variations are very useful is solving problems relating to time and work, and time and distance. Now we shall solve some more problems on time and work and on time and distance.

Example 11: A car can finish a certain journey in 10 hours at the speed of 48 kilometres per hour. By how much should its speed be increased so that it may take only 8 hours to cover the same distance?

Solution : Let the desired speed of the car in the second case be x kilometres per hour. Then the relevant table is:

Number of hours	10	8
Speed (in km per h)	48	x

We note that the greater the speed, the lesser would be the time taken. So the number of hours and speed vary inversely as each other Hence,

$$10 \times 48 = 8 \times x$$

$$\therefore \quad x = 60$$

Thus, the required increase in speed = (60 – 48) km per hour = 12 km per hour.

Example 12 : How many days will 1648 persons take to construct a bridge, if 721 persons can build the same in 48 days?

Solution : Let the desired number of days to construct the bridge be x. Then, the relevant table to construct the bridge is

Number of days	48	x
Number of persons needed	721	1648

Obviously, the more the number of persons, the quicker they will make the bridge. So the number of days and persons vary inversely as each other. Hence, we get the proportion as

$$1648 : 721 = 48 : x$$

(Writing the terms of the second ratio in the reverse way, i.e., writing $48 : x$ instead of $x : 48$)

or $$721 \times 48 = 1648 \times x$$

or $$x = \frac{721 \times 48}{1648} = 21$$

i.e., 1648 persons can finish the work in 21 days.

Example 13 : A train 315 m long is running at 54 km/*h*. How much time will it take to cross a pole?

Solution : The length of the train is given in metres. Hence, we shall convert its speed in m/second.

54 km = 54000 m

1 hour = 3600 seconds

Hence, speed = $\frac{54000}{3600}$ m / second = 15 m / second.

To cross a pole, the train will cover a distance equal to its length, namely 315 m. Let the required time be x seconds. We have the following table :

(Distance in m)	15	315
Time (in seconds)	1	x

Clearly, it is a case of direct variation.

Therefore, $$\frac{15}{1} = \frac{315}{x}$$

or $$x = \frac{315}{15} = 21$$

Hence, time taken by the train to cross the pole is 21 seconds.

Example 14 : A tempo travels at the speed of 36 km / h. How much distance will it travel in 20 seconds?

Solution : 36 km = 36000 m

1 hour = 3600 seconds

Let the required distance be x m. Thus, we have the following table :

Time (in seconds)	3600	20
Distance (in m)	36000	x

It is a case of direct variation.

$$\therefore \quad \frac{3600}{36000} = \frac{20}{x}$$

or

$$x = \frac{20 \times 36000}{3600} = 200$$

Thus, the distance travelled in 20 seconds is 200 m.

Example 15 : A train covers a certain distance in 3.5 hours at a speed of 60 km/h. How much time will the train take to cover the same distance at a speed of 80 km/h ?

Solution : Let the required time be x hours. Thus, we have the following table :

Speed (in km/h)	60	80
Time taken (in h)	3.5	x

Clearly, it is a case of inverse variation.

Therefore, $$60 \times 3.5 = 80 \times x$$

or

$$x = \frac{60 \times 3.5}{80} = \frac{21}{8} = 2\frac{5}{8}$$

Hence, the train will take $2\frac{5}{8}$ hours at a speed of 80 km/h.

EXERCISE 5.2

1. Which of the following vary inversely as each other?

(i) The number of pencils x you can buy with Rs 12.00 and the cost y per pencil (which you can assume to be a multiple of 25 paise in this case).

(ii) The number x of persons hired to construct a wall and the time y taken to finish the job

(iii) The length x of a journey by bus and the price y of the ticket.

2. u and v vary directly as each other. When u is 10, v is 15, which of the following is not a possible pair of corresponding values of u and v?

   (i) 2 and 3 (ii) 8 and 12

   (iii) 15 and 20 (iv) 25 and 37.5

3. x and y vary inversely as each other. When x is 10, y is 6 Which of the following is not a possible pair of corresponding values of x and y?

   (i) 12 and 5 (ii) 15 and 4

   (iii) 25 and 2.4 (iv) 45 and 1.3

4. A train is travelling at the average speed of 50 km/h. How much distance would it travel in 12 minutes?

5. How much time would a cyclist moving at the average speed of 6 km/h take in covering a distance of 19.5 km?

6. A car needs 10 litres of petrol for covering a distance of 135 kilometres. How much petrol is needed for the car to cover a distance of 216 km?

7. A bullock-cart covers a distance of 18 km in four hours and a half. What is the average speed of the cart?

8. Twenty pumps can empty a reservoir in 12 hours. In how many hours can 45 such pumps do the same work ?

9. If 1800 persons can finish the construction of a building in 40 days, how many persons are needed for the construction of the building in 24 days?

10. A stock of foodgrain is enough for 500 persons for 8 weeks. How long will the same stock last for 400 persons?

11. Shalu cycles to her school at an average speed of 12 km/h. It takes her 20 minutes to reach the school. If she wants to reach her school in 15 minutes, what should be her average speed?

12. A shopkeeper has just enough money to buy 52 cycles worth Rs 525.00 each. If each cycle were to cost Rs 21.00 more, then how many cycles would he be able to buy with that amount of money?

13. A contractor who had a work-force of 560 persons, undertook to complete a portion of a stadium in 9 months He was asked to complete the job in 7 months. How many extra persons had he to employ?

14. The following readings on pressure (P) and volume (V) at a constant temperature of a gas were recorded :

Pressure (in cm)	75.00	80.00	–	112.50	–
Volume (in ml)	12.00	–	10.00	–	15.00

Assuming that P varies inversely as V, find the missing terms in the above Table.

15. The following readings on volume (V) and absolute temperature (T) on a constant pressure of gas were recorded .

Volume (V)	10	12	15		
Absolute temperature (T)	300			750	375

Assuming that V varies directly as T, find the missing terms in the above Table.

16. 50 persons can do a piece of work in 18 days. In how many days can the same piece of work be done by 75 persons?

17. 6 persons can complete the electric fittings in a building in 7 days. How many days will it take if 21 persons do the same job?

18. Working 8 hours a day, Anu can copy a book in 18 days. How many hours a day should she work so as to finish the work in 12 days?

19. A 270 m long goods train is running at 40.5 km / h How much time will it take to cross a tree?

20. A train is running at 36 km / h. If it crosses a pole in 25 seconds, find its length.

21. A train 450 m long crosses a pole in $22\frac{1}{2}$ seconds. What is the speed of the train in km/h?

22. A 250 m long train is running at a speed of 55 km / h. In how much time will it cross a platform of length 520 m?

[*Hint* : Distance travelled by the train in crossing the platform = Length of the train + length of the platform.]

23. A train moving at a speed of 80 km / h covers a certain distance in 4.5 hours What should be the speed of the train to cover the same distance in 3 hours?

Things to Remember

1. Two quantities x and y are said to vary directly as each other, if they increase (decrease) together in such a manner that the ratio of their corresponding values remains constant. Equivalently, x and y vary directly, if $\frac{x}{y} = k$ where k is a positive constant number

2. Two quantities x and y are said to vary inversely as each other, if an increase in x causes a proportionate decrease in y (and vice-versa) in such a manner that the product of their corresponding values reamains constant. That is, if $xy = k$ (where k is a positive constant number), then x and y vary inversely

3. The relation between time, distance and speed is

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

PERCENTAGE AND ITS APPLICATIONS

CHAPTER 6

6.1 Introduction

In Class VI, you have studied the idea of percentage. Recall that a fraction with denominator 100 is called a *per cent*. Per cent is an abbreviation for the Latin word *per centum* meaning per hundred or hundredths and is denoted by the symbol '%'.

Thus, $\frac{10}{100} = 10 \times \frac{1}{100} = 10\%$

$$\frac{23}{50} = \frac{23\times 2}{50\times 2} = \frac{46}{100} = 46\times\frac{1}{100} = 46\%$$

We shall solve some more problems on percentage in this Chapter. We shall also use the idea of percentage in solving problems of profit and loss. Further, formula relating to simple interest, time, rate of interest and principal will be introduced and some simple problems based on the formula will be solved.

6.2 Some Problems on Percentage

In Class VI, we have already learnt how to solve some problems on percentage. Now we take some more examples to consolidate and continue that learning.

Example 1 : If 23% of a number is 46, find the number.

Solution : Let the number be x.

$\therefore$ 23% of $x = 46$

or $\frac{23}{100}\times x = 46$

or $$x = \frac{46 \times 100}{23} = 200$$

Thus, the number is 200.

Example 2 : In an examination, Neeta secured 372 marks. If she secured 62% marks, find the maximum marks.

Solution : Let the maximum marks be x.

Neeta's marks = 62% of x

Neeta secured 372 marks.

$\therefore$ $$62\% \text{ of } x = 372$$

or $$\frac{62}{100} \times x = 372$$

or $$x = \frac{372}{62} \times 100 = 600$$

Hence, maximum marks are 600.

Example 3 : In an election, a candidate got 60% of the total valid votes. 15% of the total votes were declared invalid. If the total number of votes is 500000, find the number of valid votes polled in favour of the candidate.

Solution : Total number of invalid votes = 15% of 500000

$$= \frac{15}{100} \times 500000 = 75000$$

Total number of valid votes polled = 500000 – 75000 = 425000

Percentage of valid votes polled in favour of the candidate = 60%

Therefore, the number of valid votes polled in favour of the candidate

$$= 60\% \text{ of } 425000$$

$$= \frac{60}{100} \times 425000 = 255000.$$

Thus, the number of valid votes polled in favour of the candidate is 255000.

EXERCISE 6.1

1. Find the quantity b, if

   (i) 3% of b is Rs 9 (ii) $\frac{1}{2}$% of b is Rs 50 (iii) 3.4% of b is Rs 68

2. A person deposits Rs 600 per month in his Savings Bank Account If this is 15 % of his monthly income, find his monthly income.
3. Gowang went to school for 216 days in a full year If his attendance is 90%, find the number of days on which the school was opened.
4. The annual increase in population of a town is 5% The increase in population from 1999 to 2000 is 8820. What was the population of the town in 1999 ?
5. Sudha obtained 504 marks in a certain examination. If she obtained 63% of the total marks, then find the total marks.
6. Kishan spends 30% of his salary on food and donates 3% of his salary in a Charitable Trust. In a particular month, he spent Rs 2310 on these two items. What is his total salary for this month?
7. The strength of students in a school increased by 8% If the actual increase in the number of students is 160, find the original strength of the school Also, find the new strength.
8. 60% of the students in a school are girls. If the total number of girls in the school is 690, find the total number of students in the school. Also, find the number of boys in the school.
9. A football team won 40% of the total number of matches it played during a year. If it lost 12 matches in all and no match was drawn, find the total number of matches played by the team during the year.

   [*Hint* : Matches lost = (100 – 40)% of the total matches]

10. 80% of the total number of students of a school did not go for a picnic on a certain day. If the number of students who went for the picnic was 240, find the total number of students in the school.

6.3 Profit and Loss

In earlier classes, you have learnt about profit and loss. If the selling price (S.P) of an article is more than its cost price (C.P.), we say that there is a profit. If on the other hand, the S. P. of an article is less than its C. P., we say that there is a loss. Thus,

Profit = S.P. – C.P., if S.P > C.P.
Loss = C.P. – S.P., if C.P. > S.P.

It may be kept in mind that in addition to the price paid for purchasing goods, sometimes an individual has to spend some money towards transportation, rent of the shop, etc. These are all included under the general name of *overhead charges* and form a part of the cost price of the goods purchased. Suppose that a retailer bought three dozen cricket balls for Rs 81 and paid Rs 9 as transportation charges for the same. Then the actual cost price of these three dozen cricket balls is Rs 90. *Thus, C.P. includes overhead charges.*

Suppose that a dealer buys a cooler for Rs 2200 and spends Rs 300 on its repairs. He sells it for Rs 3000. Thus, he makes a profit of Rs (3000 – 2500) = Rs 500, whereas if he sells the cooler for Rs 2200, he suffers a loss of Rs (2500 – 2200) = Rs 300.

As the profit or loss is always on the initial investment (i.e., cost price including overhead charges), it is customary to express profit or loss as a per cent of the cost price. Thus, by selling the cooler for Rs 3000, the shopkeeper makes a gain of

$$\left(\frac{500}{2500}\times 100\right)\%\text{, i.e., }20\%.$$

Similarly, by selling the cooler for Rs 2200, the shopkeeper suffers a loss of

$$\left(\frac{300}{2500}\times 100\right)\%\text{, i.e., }12\%.$$

Thus, we see that

$$\text{Profit per cent} = \frac{\text{Profit}}{\text{Cost Price}}\times 100$$

and

$$\text{Loss per cent} = \frac{\text{Loss}}{\text{Cost Price}}\times 100$$

Let us now take some examples to illustrate the above ideas.

Example 4 : A retailer buys a watch for Rs 335. By paying Rs 15 more, he replaces its strap by a costlier strap If he sells the watch for Rs 400, determine his gain or loss per cent.

Solution : C.P. of the watch with the new strap = Rs 335 + Rs 15 = Rs 350

S.P. of the watch = Rs 400

∴ Gain = Rs (400 – 350) = Rs 50 (S.P. > C.P.)

∴ $$\text{Gain per cent} = \left(\frac{50}{350}\times 100\right) = 14\frac{2}{7}$$

Example 5 : Gurdeep lost 20% by selling a bicycle for Rs 1536. Find the cost price of the bicycle.

Solution : S.P. = Rs 1536, Loss = 20%

Let C.P. = Rs 100

Loss = 20% of Rs 100 = Rs 20

∴ S.P. = Rs 100 – Rs 20 = Rs 80

If the S.P. is Rs 80, then C.P = Rs 100

∴ If the S.P. is Rs 1536, then the C.P. $= \text{Rs}\left(\frac{100}{80}\times 1536\right)$

= Rs 1920

Thus, the cost price of the bicyle is Rs 1920.

Example 6 : Devi purchased a house for Rs 452000 and spent Rs 28000 on its repairs. She had to sell it for Rs 468000. Find her profit or loss per cent.

Solution : Cost price includes the overhead charges also.

Therefore, C.P. = Rs (452000 + 28000) = Rs 480000

S.P. = Rs 468000

Since S.P. < C.P., therefore loss = C.P. – S.P.

= Rs 480000 – Rs 468000

= Rs 12000

∴ $$\text{Loss per cent} = \left(\frac{12000}{480000}\times 100\right) = \frac{5}{2}$$

Example 7 : Harish bought a second-hand typewriter for Rs 1200 and spent Rs 200 on its repairs. He sold it for Rs 1680. Find his profit or loss. What was his profit or loss per cent?

Solution : Cost price of the typewriter after repairing = Rs (1200 + 200)

= Rs 1400

S.P. = Rs 1680

Since S.P. > C.P., therefore profit = S.P. – C.P.

= Rs 1680 – Rs 1400 = Rs 280

Further, profit per cent = $\frac{\text{Profit}}{\text{C.P.}} \times 100$

$= \frac{280}{1400} \times 100 = 20$

∴ Profit = 20%.

Example 8 : By selling a coat for Rs 630, a shopkeeper gains 5%. Find the cost price of the coat.

Solution : Here S.P. = Rs 630, Gain = 5% and C.P. = ?

Let the cost price of the coat be Rs 100.

∴ Gain = Rs 5

∴ S.P. of the coat = Rs 105

If the S.P. of the coat is Rs 105, then C.P. = Rs 100

∴ If the S.P. of the coat is Re 1, then C.P. = Rs $\frac{100}{105}$

∴ If the S.P. of the coat is Rs 630, then C.P. = Rs $\frac{100}{105} \times 630$ = Rs 600

∴ The cost price of the coat is Rs 600.

EXERCISE 6.2

1. Complete the following table (wherever possible).

Purchased at	*Overhead expenses*	*Cost price*	*Selling price*	*Profit*	*Loss*	*Profit %*	*Loss %*
(i) Rs 370	Rs 80	–	–	Rs 90	–	–	–
(ii) Rs 3000	–	Rs 3100	–	–	Rs 62	–	–
(iii) Rs 28000	Rs 2000	–	Rs 36000	–	–	–	–
(iv) –	Rs 500	Rs 900	–	–	–	8	–
(v) Rs 240	Rs 10	–	–	–	–	–	6

2. A bookseller bought a book for Rs 15 and spent Rs 5 on its binding. He sold it for Rs 24. Find his gain per cent.
3. Joshi bought a second-hand car for Rs 70000 and spent Rs 5000 on overhauling and painting, etc. He then sold it for Rs 67500 Find his gain or loss per cent
4. A dealer buys a wrist watch for Rs 225 and spends Rs 15 on its repairs If he sells the same for Rs 300, find his profit per cent.
5. A retailer buys a cooler for Rs 1200 and pays Rs 40 as transportation charges If he sells the cooler for Rs 1550, determine his profit per cent
6. Karim bought 150 dozen pencils at Rs 20 a dozen. His overhead expenses were Rs 200. He sold the pencils at Rs 2 40 each. What was his profit or loss per cent?
7. By selling a table for Rs 990, a trader gains 10%. Find the cost price of the table
8. John sold his T.V. for Rs 10240, thereby suffering a loss of 20% Find the cost price of the T.V
9. By selling a bucket for Rs 240, a blacksmith loses 20% of his cost. If he sells it for Rs 360, what profit or loss would be there for him?
10. Harish bought a cycle for Rs 960 and sold it to Subramaniam at a profit of 5%. Subramaniam sold it to Mukul at a profit of 10%. Find the money paid by Mukul for the cycle.
11. If by selling a bed sheet for Rs 150, a person loses 4%, for what amount should he sell it so as to gain 20%?
12. By selling a stool for Rs 135, a carpenter loses 10% How much per cent would he gain or lose by selling it for Rs 165?

6.4 Simple Interest

You have already read about Simple Interest (S.I.) in Class VI and you have calculated it using the Unitary Method. We now develop a formula for calculating simple interest.

Let us consider the following example :

Find the interest on Rs 5000 for 3 years at the rate of 5% per annum

Here, the rate of interest is 5% per annum. It means that

Interest on Rs 100 for 1 year = Rs 5

$\therefore$ Interest on Re 1 for 1 year = Rs $\frac{5}{100}$

$\therefore$ Interest on Rs 5000 for 1 year = Rs $\frac{5000 \times 5}{100}$

$\therefore$ Interest on Rs 5000 for 3 years = Rs $\frac{5000 \times 5 \times 3}{100}$

Now let the principal be P, rate of interest R% per annum and the time T years. We may, as before, calculate the interest I as follows :

The rate of interest is R% per annum.

It means that

Interest on Rs 100 for 1 year = Rs R

$\therefore$ Interest on Re 1 for 1 year = Rs $\frac{R}{100}$

$\therefore$ Interest on Rs P for 1 year = Rs $\frac{P \times R}{100}$

$\therefore$ Interest I on Rs P for T years = $\frac{P \times R \times T}{100}$

Thus,

$$I = \frac{P \times R \times T}{100}$$

From the above formula, we observe that

$$P = \frac{100 \times I}{R \times T}\ ;\ R = \frac{100 \times I}{P \times T} \text{ and } T = \frac{100 \times I}{P \times R}.$$

We shall now solve some problems using the above formulae.

Example 9 : Find S.I. on Rs 5000 at 12% per annum for 3 years.

Solution : P = Rs 5000, R = 12, T = 3 years, I = ?

$$\therefore \quad I = \frac{P \times R \times T}{100}$$

$$= \text{Rs } \frac{5000 \times 12 \times 3}{100} = \text{Rs } 1800$$

Example 10 : Find :

(i) S.I. on Rs 5664 at $13\frac{3}{4}\%$ per annum for 9 months.

(ii) S.I. on Rs 3125 at 15% per annum for 73 days.

Solution : (i) P = Rs 5664, $R = 13\frac{3}{4} = \frac{55}{4}$, $T = \left(\frac{9}{12}\right)$ year $= \left(\frac{3}{4}\right)$ year

We have

$$I = \frac{P \times R \times T}{100}$$

$$= \text{Rs}\left(\frac{5664 \times \frac{55}{4} \times \frac{3}{4}}{100}\right)$$

$$= \text{Rs}\left(5664 \times \frac{55}{4} \times \frac{3}{4} \times \frac{1}{100}\right) = \text{Rs } 584.10$$

(ii) P = Rs 3125, R = 15, $T = \left(\frac{73}{365}\right)$ year $= \left(\frac{1}{5}\right)$ year

$$\therefore \qquad I = \frac{P \times R \times T}{100} = \text{Rs}\left(\frac{3125 \times 15 \times \frac{1}{5}}{100}\right) = \text{Rs } 93.75$$

Example 11 : Find at what rate of interest per annum will Rs 500 amount to Rs 605 in 3 years.

Solution : Denoting amount by A, P = Rs 500, A = Rs 605, T = 3 years, R = ?

We know that A = P + I.

$$\therefore \qquad I = A - P = \text{Rs } 605 - \text{Rs } 500 = \text{Rs } 105$$

$$\therefore \qquad R = \frac{100\times I}{P\times T} = \frac{100\times 105}{500\times 3} = 7$$

Hence, the rate of interest is 7% per annum.

Example 12 : In how many years do Rs 250 invested at the rate of 8% per annum simple interest amount to Rs 330?

Solution : Here P = Rs 250, A = Rs 330, R = 8, T = ?

$$\therefore \qquad I = \text{Rs } (330 - 250) = \text{Rs } 80$$

$$\therefore \qquad T = \frac{100\times I}{P\times R} = \frac{100\times 80}{250\times 8} = 4$$

Thus, the required time is 4 years.

Example 13 : A sum amounts to Rs 2437.50 at $12\frac{1}{2}$% simple interest per annum after 4 years. Find the sum.

Solution : A = Rs 2437.50, R = $12\frac{1}{2} = \frac{25}{2}$, T = 4 years, P = ?

Let the sum be Rs x.

$$\text{Then} \qquad \text{S.I.} = \text{Rs}\left(x\times\frac{25}{2}\times 4\times\frac{1}{100}\right) = \text{Rs } \frac{x}{2}$$

$$\therefore \qquad \text{Amount} = \text{Rs}\left(x+\frac{x}{2}\right) = \text{Rs}\left(\frac{3x}{2}\right)$$

$$\text{Thus} \qquad \frac{3x}{2} = 2437.50$$

$$\text{or} \qquad x = \frac{2437.50\times 2}{3} = 1625$$

Hence, the sum is Rs 1625.

Example 14 : In how many years will a sum of Rs 400 yield an interest of Rs 112 at 14% per annum?

Solution : Here P = Rs 400, I = Rs 112, R = 14, T = ?

$$\therefore \qquad T = \frac{I \times 100}{P \times R} = \frac{112 \times 100}{400 \times 14} = 2$$

Thus, in two years, a sum of Rs 400 will yield an interest of Rs 112 at 14% per annum

Check : P = Rs 400, T = 2 years, R = 14

$$I = \frac{P \times R \times T}{100} = \text{Rs } \frac{400 \times 2 \times 14}{100} = \text{Rs } 112,$$

which is as given in the problem.

Example 15 : Simple interest on a certain sum is $\frac{16}{25}$ of the sum. Find the rate per cent and time, if both are numerically equal.

Solution : Let the sum = Rs x. Then S.I. = Rs $\frac{16}{25}x$, R = ?, T = ?

Let rate = R% per annum and Time = R years.

$$\text{Now,} \qquad \text{S.I.} = \frac{P \times R \times T}{100}$$

$$\therefore \qquad \frac{16}{25}x = \frac{x \times R \times R}{100}$$

$$\text{or} \qquad \frac{R^2}{100} = \frac{16}{25}$$

$$\text{or} \qquad R^2 = \frac{1600}{25} = \left(\frac{40}{5}\right)^2$$

$$\therefore \qquad R = \frac{40}{5} = 8$$

Hence, rate = 8% and time = 8 years.

EXERCISE 6.3

1. Find the unknown quantity in each of the following :
   (i) Principal = Rs 500, Rate of interest per annum = 12%, Time period = 3 years, Interest = ...

(ii) Principal = Rs 1250, Rate of interest per annum = 14%, Time period = 4 years, Interest = ...

(iii) Principal = Rs 200, Time period = 5 years, Rate of interest per annum = 6%, Interest = ..., Amount = ...

(iv) Principal = Rs 600, Rate of interest per annum = 10%, Time period = $3\frac{1}{2}$ years, Interest = ..., Amount = ...

(v) Principal = Rs 280, Rate of interest per annum = 4%, Time period = $2\frac{1}{2}$ years, Interest = ...

(vi) Principal = Rs 2850, Rate of interest per annum = $3\frac{1}{2}$ %, Time period = 8 months, Interest = ...

(vii) Principal = Rs 180, Rate of interest per annum = 3%, Time period = $1\frac{1}{4}$ years, Interest = ..., Amount = ...

(viii) Principal = Rs 560, Time = 73 days, Rate of interest per annum = ..., Interest = Rs 14.

(ix) Principal = Rs 480, Time period = 8 months, Rate of interest = $2\frac{1}{2}$ % per annum, Interest = ..., Amount = ...

(x) Principal = Rs 720, Rate of interest per annum = 4%, Time period = ..., Interest = Rs 72, Amount = ...

(xi) Principal = Rs 500, Time period = 3 years, Rate of interest per annum = ..., Interest = ..., Amount = Rs 650.

2. Find simple interest when

(i) Principal = Rs 800, Rate = 6% per annum and Time = 4 years.
(ii) Principal = Rs 450, Rate = 12% per annum and Time = 3 years.
(iii) Principal = Rs 600, Rate = 2% per annum and Time = 20 months.

Also, find the amount in each case.

3. Find principal when
   - (i) Simple interest = Rs 36, Rate = 3% per annum and Time = 3 years.
   - (ii) Simple interest = Rs 140, Rate = 16% per annum and Time = $2\frac{1}{2}$ years.
   - (iii) Simple interest = Rs 72, Rate = 3% per annum and Time = 3 months.
4. Find time when
   - (i) Principal = Rs 1000, Rate = 8% per annum and Simple interest = Rs 200.
   - (ii) Principal = Rs 640, Rate = $12\frac{1}{2}$ % per annum and Simple interest = Rs 40.
   - (iii) Principal = Rs 10000, Rate = 18% per annum and Simple interest = Rs 12600.
5. Find rate when
   - (i) Principal = Rs 500, Simple interest = Rs 150 and Time = 4 years.
   - (ii) Principal = Rs 400, Simple interest = Rs 78 and time = $1\frac{1}{2}$ years.
   - (iii) Principal = Rs 700, Simple interest = Rs 168 and Time = 16 months
6. Anita deposits Rs 1000 in a savings bank account. The bank pays interest at the rate of 5% per annum. What amount can Anita get after one year?
7. Veena deposited Rs 7200 in a finance company which pays 15% interest per year. Find the amount she is expected to get after $4\frac{1}{2}$ years.
8. William deposits Rs 520 in a bank. The bank pays interest at 8% per annum. Find the interest and the amount to be received by William after two years.
9. Kishan Lal borrowed a sum of money at 9% per annum. If he paid Rs 594 as interest after three years, find the sum borrowed by him.
10. Akhtar borrowed a sum of money at $10\frac{1}{2}$ % per annum from a bank. If he paid Rs 1863 75 as interest for $2\frac{1}{2}$ years, find the sum.
11. A sum of money amounted to Rs 2520 at 10% per annum in 4 years. Find the sum.

12. A sum of money amounted to Rs 2040 in $2\frac{1}{2}$ years at 11% per annum. Find the sum.

13. A sum of money becomes $\frac{7}{4}$ of itself in 6 years at a certain rate of interest. Find the rate of interest.

14. Hamid borrowed Rs 1500 from a financer. After $3\frac{1}{2}$ years, he paid Rs 2655 to the financer and cleared the accounts. Find the rate of interest charged.

15. In how much time will a sum become double of itself at $12\frac{1}{2}$% per annum simple interest?

[*Hint* · Let principal be P, then amount = 2P.]

Things to Remember

1. Profit = S.P. – C.P. (If S.P > C P.)
Loss = C P. – S.P (If S.P < C.P.)

2. Overhead charges are included in the cost price.

3. $\text{Profit }\% = \frac{\text{Profit}}{\text{C P.}} \times 100$

$\text{Loss }\% = \frac{\text{Loss}}{\text{C.P.}} \times 100$

4. Simple Interest $I = \frac{P \times R \times T}{100}$, where P = Principal, R % = Rate of interest per annum, T = Time

5. $P = \frac{I \times 100}{R \times T}$, $R = \frac{I \times 100}{P \times T}$ and $T = \frac{I \times 100}{P \times R}$, where I, P, R and T have their usual meanings.

6. Amount = Principal + Interest
or $A = P + I$

As History Tells Us

The history of commercial activities is as old as human civilization. With the development of civilization, people learnt to live in a community and the barter system came into use. This promoted the idea of commerce and trade. Subsequently, 'money' also came into use to make this exchange easier and thus trade flourished From the clay tablets of the period 2450 BC to 2330 BC, it is evident that the Babylonians were familiar with bills, promissory notes, mortgages, taxes, simple and compound interests and other commercial activities.

The Indian mathematicians *Brahmagupta and Bhaskara* are well known for giving the idea of the *Rule of Three* which was highly regarded by merchants as a tool for centuries. Another Indian mathematician *Mahavira* (around 850 AD) also stated this rule in nearly the same form.

The concepts of direct and inverse variations were also known. The following problem is given in Bhaskara's *Lilawati*

Bullocks which have plowed four seasons cost four niskas (निष्क). *What will bullocks which have plowed twelve seasons cost ?*

The Romans made use of fractions which easily reduce to hundredths without recognizing per cents as such. In the Italian manuscripts of commercial mathematics of the fifteenth century, it is common to find expressions such as '20 p 100', 'x p cento' and 'vi p c^{o}' for 20%, 10% and 6% respectively. Thus, the per cent sign was initially written as 'per c^{o}', 'p c^{o}', etc In the middle of the seventeenth century, it developed into the form 'per ÷'. Finally, 'per' was dropped and it took the present form '%'.

The Indian mathematician *Bhaskara* has used per cents in the interest problems in his famous book *Lilawati* In the sixteenth century, per cents were mainly used for the computation of interest, and profit and loss. The phrase 'profit and loss' had been in use in the same sense as we use it today The popularity of this topic in the sixteenth century can be inferred from the fact that mathematician *Werner* devoted forty-seven pages to it in his book *Rechenbuch* (around 1561 AD).

ALGEBRAIC EXPRESSIONS

CHAPTER 7

7.1 Introduction

In the previous Class, you studied algebraic expressions. As you know, *algebraic expressions* are obtained when we combine numbers and literal numbers by means of arithmetic operations. You also know how to add two algebraic expressions and how to subtract one algebraic expression from another. Recall that algebraic expressions containing one, two and three terms are respectively called *monomials, binomials* and *trinomials.*

In this Chapter, we shall tell you how to multiply algebraic expressions of the above types. This will enable you to solve numerous problems of daily life. In particular, you will be able to evaluate certain types of numeric expressions with great ease. You will also be able to solve problems related with areas of rectangles and volumes of cuboids.

We shall also tell you about some simple algebraic relations called *identities.* Some interesting activities will be carried out regarding these identities. Throughout this Chapter, an 'expression' will mean an 'algebraic expression'.

7.2 Multiplication of Monomials

Recall that a monomial is a one-term expression that is either a constant (number) or a literal or the product of constants and literals having an integral exponent (power). However, *a monomial cannot contain the operations of addition and subtraction.* '+' and '–' may only occur at the left of a monomial as the sign of the constants.

For example,

10, –7, $2x$(or $+2x$), $-3x$, $4ab^2$, $-15s^2t$, $506\ x^2y^2z^3$, $100\ xy^{-1}$, $6\ a^{-2}b^2c^{-3}d$, etc. are monomials. We generally refer to monomials like 10 and –7 as *constants.* Also, we generally take the exponents to be positive integers. Thus, we shall not consider expressions like $100xy^{-1}$, etc. in this book.

Recall that while multiplying several numbers, we can rearrange these numbers as we like. We can also fix the order of multiplication as it suits us. For example,

$$\begin{aligned}(5 \times 91) \times 2 &= (91 \times 5) \times 2 && \text{[Rearranging terms]}\\ &= 91 \times (5 \times 2) && \text{[Multiplying terms in different order]}\\ &= 91 \times 10 = 910\end{aligned}$$

Since literals stand for numbers only, we may follow this practice in multiplying algebraic expressions also.

Let us begin with multiplying two monomials that have only one and the same literal, e.g , $2x$ and $3x^2$. Now

$$\begin{aligned}(2x) \times (3x^2) &= 2 \times x \times 3 \times x^2\\ &= (2 \times 3) \times (x \times x^2)\\ &= 6 \times x^{1+2} && [x = x^1]\\ &= 6x^3\end{aligned}$$

Remarks : 1. We rearranged and associated terms to put together numerical factors into one group (2×3) and the literals into another group $(x \times x^2)$. When there are more than one literal, we collect like literals into separate groups.

2. To simplify the group of literal $(x \times x^2)$, we used the following important result :

$$\boxed{x^m \times x^n = x^{m+n}}$$

According to this relation, *to multiply literals and literal powers with the same base, we add the exponents.*

Let us now multiply the monomials $2xy$ and $3y$ that contain two literals x and y.

$$\begin{aligned}(2xy) \times (3y) &= 2 \times x \times y \times 3 \times y\\ &= (2 \times 3) \times (x) \times (y \times y)\\ &= 6 \times x \times y^{1+1}\\ &= 6 \times x \times y^2\\ &= 6xy^2\end{aligned}$$

Observe that the coefficient (6) of the product is the product of the coefficients (2 and 3) of the monomials being multiplied.

Let us take another example :

$$\begin{aligned}(3ab^2) \times (4a^2b^2) &= 3 \times a \times b^2 \times 4 \times a^2 \times b^2\\ &= (3 \times 4) \times (a \times a^2) \times (b^2 \times b^2)\end{aligned}$$

$$= 12 \times (a^3) \times (b^4) \quad \text{[Adding exponents in each group of literals]}$$

$$= 12a^3b^4$$

Observe that, in this case also, the coefficient of the product is the product of the coefficients of the monomials being multiplied. As a matter of fact, we have the following *thumb-rule* for multiplying any number of monomials :

(a) The coefficient of the product of given monomials is the product of the coefficients of these monomials.

(b) The literal part of the product contains all the literals occurring in the given monomials; the exponent of each literal is the sum of the exponents of this *literal* in the given monomials.

Example 1 : Multiply the monomials $3x^2y^3$ and $-7xy^2z$.

Solution : Using the above thumb-rule,

$$(3x^2y^3) \times (-7xy^2z) = \{3 \times (-7)\} \times (x^{2+1}) \times (y^{3+2}) \times (z)$$
$$= -21\, x^3y^5z$$

Example 2 : Find the product of $-7pqr$, $3p^2q$ and $-2pr^2$. Verify your result for $p = 1$, $q = 2$ and $r = 3$.

Solution : Using the thumb-rule given above,

$$(-7pqr) \times (3p^2q) \times (-2pr^2) = \{(-7) \times (3) \times (-2)\} \times p^{1+2+1} \times q^{1+1} \times r^{1+2}$$
$$= 42 \times p^4 \times q^2 \times r^3$$
$$= 42\, p^4\, q^2\, r^3$$

For $p = 1$, $q = 2$ and $r = 3$,

$$\text{LHS} = (-7 \times 1 \times 2 \times 3) \times (3 \times 1^2 \times 2) \times (-2 \times 1 \times 3^2)$$
$$= (-42) \times 6 \times (-18)$$
$$= (-252) \times (-18)$$
$$= 252 \times 18$$
$$= 4536$$
$$\text{RHS} = 42 \times 1^4 \times 2^2 \times 3^3$$
$$= 42 \times 4 \times 27$$
$$= 168 \times 27$$
$$= 4536$$

Hence, LHS = RHS.

Remarks: 1. If for particular values of the variables, LHS ≠ RHS, it is certain that you have made a mistake. Therefore, it is always a good idea to verify your result for small non-zero values of the variables.

2. Recall that the area of a rectangle with length l and breadth b is $l \times b$ If we think of l and b as monomials, then the product of two monomials may be regarded as the area of a rectangle. Thus, the product $21a^2$ of the monomials $3a$ and $7a$ may be regarded as the area of a rectangle with length $7a$ and breadth $3a$ (Fig.7.1) This observation helps us in understanding some algebraic identities.

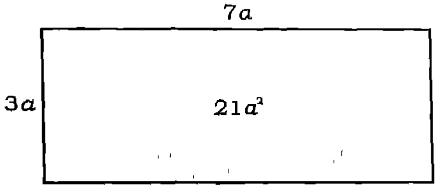


Fig 7.1

7.3 Multiplication of a Monomial and a Binomial

Recall that a *binomial* is a sum or difference of *two* monomials. Thus $x + y$, $2x - 3y$, $a^2b + 7c$, $9p - \frac{7}{2}$, $2.3q - 1.9r$ etc. are binomials. To multiply a monomial and a binomial, we have to evaluate expressions like

$$9x\,(6y + 10z),\ a^2bc\,(2c - 3ac^3),\ \text{etc.}$$

Since the literals stand for some numbers only, these expressions are like $a(b + c)$ where a, b and c are numbers. You know that

$$a\,(b + c) = ab + ac,\ a(b - c) = ab - ac \qquad (1)$$

$$(a + b)\,c = ac + bc,\ (a - b)\,c = ac - bc \qquad (2)$$

So, to multiply a monomial and a binomial, we use the properties

$$\text{P}(\text{Q} + \text{R}) = \text{PQ} + \text{PR},\ \text{P}(\text{Q} - \text{R}) = \text{PQ} - \text{PR} \qquad (3)$$

$$(\text{P} + \text{Q})\text{R} = \text{PR} + \text{QR},\ (\text{P} - \text{Q})\,\text{R} = \text{PR} - \text{QR} \qquad (4)$$

where P, Q and R are monomials (and so P + Q and P – Q are binomials).

Example 3: Multiply $\frac{1}{2}a^2b$ and $3ab - 5a^3b^4$.

Solution : Using relation (3) above,

$$\frac{1}{2}a^2b(3ab - 5a^3b^4) = \frac{1}{2}a^2b \times 3ab - \frac{1}{2}a^2b \times 5a^3b^4$$

$$= \left(\frac{1}{2}\times 3\right)\times a^{2+1}\times b^{1+1} - \left(\frac{1}{2}\times 5\right)\times a^{2+3}\times b^{1+4}$$

$$= \frac{3}{2}a^3b^2 - \frac{5}{2}a^5b^5$$

Check : Let us check the result for $a = 2$, $b = 1$.

$$\text{LHS} = \frac{1}{2}a^2b(3ab - 5a^3b^4) = \frac{1}{2}\times 2^2 \times 1\ (3 \times 2 \times 1 - 5 \times 2^3 \times 1^4)$$

$$= 2(6 - 40) = -68$$

$$\text{RHS} = \frac{3}{2}a^3b^2 - \frac{5}{2}a^5b^5 = \frac{3}{2}\times 2^3 \times 1^2 - \frac{5}{2}\times 2^5 \times 1^5$$

$$= 12 - 80 = -68$$

Thus, LHS = RHS. The result is correct.

Note: The above method of multiplying a monomial with a binomial is known as the *horizontal method* or *row method*, because the working is done horizontally in rows (parallel to the lines of the page). If we arrange our working *vertically*, that is, from top to bottom in columns, the method will be called the *vertical method* or the *column method.*

Example 4 : Multiply $2x$ and $3x + 4y$ in two different ways and verify the result.

Solution : *Horizontal Method of Multiplication* :

$$2x\,(3x + 4y) = 2x \times 3x + 2x \times 4y$$

$$= 6x^2 + 8xy$$

Vertical Method of Multiplication :

$$\begin{array}{rcl} & & 3x + 4y \\ & \times & \quad 2x \\ \hline 2x\times 3x \rightarrow & & \underline{6x^2 + 8xy} \leftarrow 2x\times 4y \end{array}$$

Thus, the product turns out to be $6x^2 + 8xy$ by both the methods.

Remark : You may use any of the two methods above for multiplication; the result is the same.

Geometric Interpretation of Product of a Monomial and a Binomial

For the sake of simplicity, let us consider a monomial k and a binomial $x + y$. Let us draw a rectangle ABCD with breadth k and length $x + y$ (Fig. 7.2). Take a point P on AB such that

$$AP = x \text{ (and so } PB = y)$$

Draw PQ||AD meeting DC in Q.

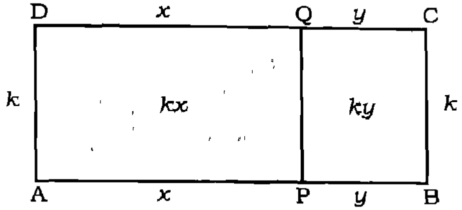


Fig. 7.2

Since rectangle ABCD = rectangle APQD + rectangle PBCQ,

$\therefore$ area of rectangle ABCD = area of rectangle APQD + area of rectangle PBCQ

$$= k \times x + k \times y$$
$$= kx + ky$$
$$= k(x + y)$$

Thus, the product $k(x + y)$ represents the area of the rectangle with breadth k and length $x + y$.

As another example, let us consider the product $2x\,(3x + 4y)$ from example 4 above. We have seen that this product is equal to $6x^2 + 8xy$. Let us now draw a rectangle ABCD with length and breadth as $3x + 4y$ and $2x$ respectively and verify that its area is $6x^2 + 8xy$ (Fig. 7.3). Take a point P on AB such that $AP = 3x$. Then $PB = 4y$.

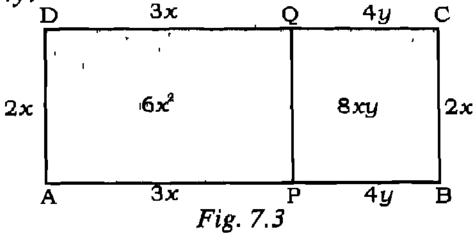


Fig. 7.3

Draw a line through P, parallel to AD, meeting DC in Q. Then

area of rectangle ABCD = area of rectangle APQD + area of rectangle PBCQ

$$= 3x \times 2x + 4y \times 2x$$
$$= 6x^2 + 8xy$$

This verifies the result.

EXERCISE 7.1

1. Multiply the monomials.

(i) $2a$ and $3a$ (ii) $2a^3$ and $4a^2$

(iii) $6ab$ and $-7bc$ (iv) $7x^2$ and $-7xyz$

(v) $\frac{2}{5}x^2y^3$ and $\frac{10}{17}xy^2$ (vi) $\frac{3}{4}abc$ and $\frac{8}{9}a^2b^3c^4$

(vii) $1.2\,pq^2$ and $0.6\,p^2q^2$ (viii) $0.9\,pqr$ and 11.0

2. Find the product :

(i) $(5a^2b)\,(3b^2c)\,(4ac^2)$

(ii) $(15pq)\,(2p^2q^2)\,(10)$

(iii) $(-3)\,(-5bc)\,(7\,b^2c^2d^2)$

(iv) $\left(\frac{2}{3}xyz\right)\left(\frac{3}{4}x^2y^2z^2\right)\left(\frac{4}{5}x^3y^3z^3\right)$

(v) $(1.1\,pq)\,(2.2\,qr)\,(3.3\,rp)$

(vi) $(0.9\,ab)\,(-0.3\,b^2c^3)\,(-2.0\,a^3c^3)$

3. Express each of the following products as a monomial :

(i) $(a^2)\,(a^{22})\,(a^{36})\,(a^{40})$

(ii) $(a^{50}b^{51})\,(b^{49}c^{67})\,(c^{33}d)\,(d^{100}a^{100})$

(iii) $\left(\frac{2}{3}ab^2c\right)\left(\frac{-9}{10}a^2\right)\left(\frac{10}{27}bc^2\right)(0.5)$

(iv) $(a^{1000})\,(b^{9999})\,(abc)\,(0)$

4. Multiply the monomials a^3, $\frac{1}{2}a^2$ and $-100a$. Verify your result for $a = -1$.

5. Multiply $0.3\,xy$ and $-100\,x^2y^3$. Verify your result for $x = 0.1$ and $y = -10$.

6. Find the value of $(32a^6)\,(-100ab^2)\,(0.5a^3b^3)$ at $a = 1$, $b = \frac{1}{2}$ after expressing it as a monomial.

7. Evaluate the product of $6.4x^3$, $8.0y^3$ and $-1.6x^2y^2$ when $x = 1.0$ and $y = 0.5$.

8. Verify that the following relations are true by expressing each side as a monomial :
   (i) $(xy)\,(x^9y^9) = (x^9y^9)\,(xy)$
   (ii) $(5abc)\left(-\dfrac{1}{500}a^5b^{50}c^{500}\right) = \left(\dfrac{-1}{500}a^5b^{50}c^{500}\right)(5abc)$

9. Simplify :
   (i) $(-3a) \times (-4a^2x^2) \times (5.5x^3)$
   (ii) $\left(\dfrac{3}{4}p^2qr\right) \times \left(5pq^2\right) \times \left(\dfrac{-8}{150}r^2\right)$

10. Find two monomials, with positive integer coefficients, whose product is the given monomial.
    (i) xyz
    [*Hint* (i) : x and yz; y and xz ; z and xy; 1 and xyz.]
    (ii) a^2b (iii) $7pq$

11. What do the following products represent geometrically ?
    (i) $3x \times 4x$ (ii) $7 \times 5p$

12. Find the numerical coefficient in the product of :
    (i) $14abcd$, $10b^2c^2$ and $2a^3d^3$
    (ii) $-3a^5b^6c^{17}$, $-4b^5c^7d^{11}$ and $5abcd$.

13. Find the literal part in the product of :
    (i) $-3.998a^2$, $-171.47\,b^2$, $36.01c^2$ and $2d^2$
    [*Hint :* Come on ! It would be the joke of the day if you were thinking of multiplying the numerical coefficients]
    (ii) $\dfrac{81}{347}xyz$, $\dfrac{1573}{410}\,y^2z^2$ and $\dfrac{810}{327}x^3z^3$.

14. Multiply :
    (i) $5a + 6$ by $3a$ (ii) $3a + 13$ by $-2a^2$
    (iii) $2x^2 - 3xy^2$ by $4x$ (iv) $-3p^2 - 7pq$ by $5pqr^2$

15. Find the product of :
    (i) $\dfrac{1}{2}x$ and $\left(\dfrac{3}{4}xy^2 + x^2y\right)$

(ii) $3ab^2$ and $\left(\frac{7}{9}a^2b^3 - \frac{2}{3}a^3b^2\right)$

16. Express as a binomial and then evaluate at $a = 2$ and $b = 1$:

(i) $(a^2b - 0.5ab^2) \times (3.3a)$ (ii) $-2.7a^2 (0.3b^2 - 0.4a^2)$

17. Multiply and verify your result for $x = 2, y = 1$ and $z = -1$:

(i) $(x^2 - y^2)(-3xy)$ (ii) $\frac{1}{2}x^3y^3z^3 (x^2 + y^2)$

18. Simplify :

(i) $a(a - b) + b(a - b)$

(ii) $a^2 - b^2 + a(a + b)$

(iii) $a(a^2 + 1) + b(b^2 + 1) - (a + b)$

(iv) $10p^2 - 6p (p + 9) + p(3 - 7p)$

7.4 Multiplication of Binomials

Once you know how to multiply a binomial with a monomial, multiplying a binomial with a binomial is easy. Recall that, given the numbers *a, b, c* and *d*, we evaluate $(a + b)(c + d)$ as follows :

$$(a + b)(c + d) = a(c + d) + b(c + d)$$

or

$$(a + b)(c + d) = (a + b)c + (a + b)d$$

If you replace each of *a, b, c* and *d* above by a monomial, you have got the rule for multiplying a binomial with a binomial. All you have to do is multiply each monomial of the first factor with each monomial of the second factor, and add all these products. This, for obvious reasons, is the *horizontal method* of multiplying two binomials.

Example 5 : Find the product $(3a + 2b)(2a - 3b)$ and verify the result for $a = -1, b = 2$.

Solution : $(3a + 2b)(2a - 3b) = 3a(2a - 3b) + 2b(2a - 3b)$

$= 6a^2 - 9ab + 4ab - 6b^2$

$= 6a^2 - 5ab - 6b^2$ [Combining the like terms $-9ab$ and $4ab$]

$\therefore \quad (3a + 2b)(2a - 3b) = 6a^2 - 5ab - 6b^2$

Verification : For $a = -1$ and $b = 2$,

LHS $= (3a + 2b)(2a - 3b)$

$$
\begin{aligned}
&= \{3 \times (-1) + 2 \times 2\}\ \{2 \times (-1) - 3 \times 2\} \\
&= (-3 + 4)\ (-2 - 6) \\
&= 1(-8) = -8 \\
\text{RHS} &= 6a^2 - 5ab - 6b^2 \\
&= 6(-1)^2 - 5 \times (-1) \times 2 - 6(2)^2 \\
&= 6 + 10 - 24 = -8
\end{aligned}
$$

Thus, LHS = RHS

Hence, the result is correct.

Note : When the two binomials being multiplied have terms with common bases as in the above example, then the product has some like terms. In such a case, it is a good idea to use the vertical method of multiplication as shown in the next example.

Example 6 : Multiply $2a^2 - ab$ and $3a + 2b$.

Solution : Since both the binomials have bases a and b in the terms, we shall use the vertical method of multiplication. It is quite like the method you have been using for multiplying two numbers say 347 and 23.

$$
\begin{array}{rcl}
347 & & \\
\times \quad 23 & \longrightarrow & 3 + 20 \\
\hline
1041 & \longrightarrow & 347 \times 3 \\
+\ 6940 & \longrightarrow & 347 \times 20 \\
\hline
7981 & & \\
\hline
\end{array}
$$

You write 23 as 3 + 20; multiply 347 by 3 first and then by 20. You write the products one below the other and then add the same.

We shall carry out the multiplication of $2a^2 - ab$ and $3a + 2b$ in exactly this way. We shall first multiply $(2a^2 - ab)$ with $3a$, and then with $2b$. We shall write the two products one below the other. You used to write units below units, tens below tens and so on. In other words, you used to write *like terms* one below the other, in columns. We shall do the same here. Finally, we shall sum the two products.

$$
\begin{array}{rrrl}
 & 2a^2 & -\ ab & \\
\times & 3a & +\ 2b & \\
\hline
 & 4a^2b & -\ 2ab^2 & \rightarrow \text{multiplying } 2a^2 - ab \text{ by } 2b \\
6a^3 & -\ 3a^2b & & \rightarrow \text{multiplying } 2a^2 - ab \text{ by } 3a \text{ and writing like terms one below the other column-wise} \\
\hline
6a^3 & +\ a^2b & -\ 2ab^2 & \rightarrow \text{adding the two products above} \\
\hline
\end{array}
$$

Alternate solution (Horizontal method) .

$$\begin{aligned}(2a^2 - ab)(3a + 2b) &= 2a^2(3a + 2b) - ab(3a + 2b)\\ &= 6a^3 + 4a^2b - 3a^2b - 2ab^2\\ &= 6a^3 + a^2b - 2ab^2\end{aligned}$$

Remarks : 1. Since the product of two binomials does not depend upon the method used, you may use any of the two methods.

2. If P, Q, R and S are monomials, then the product (P + Q) (R + S) may be regarded as the area of a rectangle with sides P + Q and R + S as shown below :

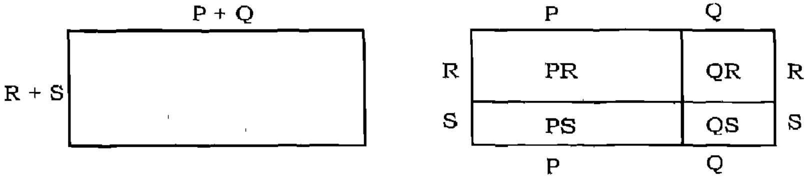


Fig. 7.4

7.5 Multiplication of a Binomial and a Trinomial

A *trinomial* is an algebraic expression containing three terms. Thus, $x^2 + y^2 + z^2$, $2x^2 + 3x^2y + 9$, $-3abc + a^3 + 10$ are trinomials. If you add three monomials with *unlike* terms, you will get a trinomial. If you add a binomial and a monomial with unlike terms, you would again get a trinomial.

Multiplying a binomial with a trinomial is quite easy if we remember the following rule for multiplying two binomials :

"To get the product of two binomials, carry out the following steps :

Step 1: *Multiply each term of the first factor with each term of the second factor. (You thus get 4 products).*

Step 2: *Add all the products obtained in Step 1.*"

The above rule also applies for the product of a binomial and a trinomial. *Multiply each term of the binomial with every term of the trinomial and add all the products so obtained.*

The only difference is that instead of four terms, you will have six terms in the product. For example, the product of $(a + b)$ and $(x + y + z)$ has the six terms

$$ax,\ ay,\ az,\ bx,\ by,\ bz$$

Thus, $$(a + b)(x + y + z) = ax + ay + az + bx + by + bz \qquad (1)$$

We may organize our working as follows :

$$\begin{aligned}(a+b)(x+y+z) &= (a+b)x + (a+b)y + (a+b)z\\ &= (ax+bx) + (ay+by) + (az+bz)\\ &= ax+bx+ay+by+az+bz,\end{aligned}$$

which can be rearranged in the form of (1) above Alternately, we may proceed as follows .

$$\begin{aligned}(a+b)(x+y+z) &= a(x+y+z) + b(x+y+z)\\ &= (ax+ay+az) + (bx+by+bz)\\ &= ax+ay+az+bx+by+bz\end{aligned}$$

This is the same as (1). We shall illustrate the three methods by examples.

Example 7 : Multiply $a^2 + b^2$ and $x^3 - y^3 + z^3$.

Solution : *Step 1* : Let us take the first term a^2 of the binomial and multiply it with every term of the trinomial. This gives us the three products

$$a^2x^3,\ -a^2y^3 \text{ and } a^2z^3$$

Next, let us take the second term b^2 of the binomial and multiply it with all the three terms of the trinomial. This gives us the three products

$$b^2x^3,\ -b^2y^3 \text{ and } b^2z^3$$

Step 2 : Adding all the products, we get

$$a^2x^3 - a^2y^3 + a^2z^3 + b^2x^3 - b^2y^3 + b^2z^3$$

Hence, $(a^2 + b^2)(x^3 - y^3 + z^3) = a^2x^3 - a^2y^3 + a^2z^3 + b^2x^3 - b^2y^3 + b^2z^3$

Alternate Solution : The argument in the above solution could be neatly expressed as follows :

$$\begin{aligned}(a^2+b^2)(x^3-y^3+z^3) &= a^2(x^3-y^3+z^3) + b^2(x^3-y^3+z^3)\\ &= a^2x^3 - a^2y^3 + a^2z^3 + b^2x^3 - b^2y^3 + b^2z^3\end{aligned}$$

Example 8 : Find the product $(x - y)(x^2 + xy + y^2)$.

Solution : Let us organize the working as follows :

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= (x-y)x^2 + (x-y)xy + (x-y)y^2\\ &= (x^3 - x^2y) + (x^2y - xy^2) + (xy^2 - y^3)\\ &= x^3 + (-x^2y + x^2y) + (-xy^2 + xy^2) - y^3\\ &\qquad \text{[Grouping like terms in decreasing powers of } x\text{]}\\ &= \underline{x^3 - y^3}\end{aligned}$$

EXERCISE 7.2

1. Multiply :
   - (i) $2x + 9$ and $6x + 5$
   - (ii) $x - 8$ and $3x + 7$
   - (iii) $\left(\frac{3}{4}a^2 + 7b\right)$ and $\left(a^3 + \frac{2}{9}b^2\right)$
   - (iv) $(2.5a + 2.3b)$ and $(2.5a - 2.3b)$
   - (v) $(2pq + 3q^2)$ and $(3pq + 2q^2)$
2. Multiply and verify the result for given values :
   - (i) $(2x - 5)(7 + 4x)$, $x = 2$
   - (ii) $(x + y)(7x - y)$, $x = 1$, $y = 0$
   - (iii) $(a^2 + b)(b^2 + a)$, $a = -1$, $b = -2$
   - (iv) $(p^2 - q^2)(p - q)$, $p = 2$, $q = 0$
3. Express in the form of a single algebraic expression :
   - (i) $(2x + 3y)(4x^2y + 5xy^2)$
   - (ii) $(a^5 + 5)(b^3 + 3) + 4$
   - (iii) $(a + bcd)(a^3 + b^3c^3d^3)$
   - (iv) $(m^2 - 2n)(-3m - 4n^2) + 3m^3$
   - (v) $(t^2 + s^3)(t^2 - s^3)$
   - (vi) $(a + b)(c - d) + (c + d)(a - b) + 2(ac + bd)$
4. Simplify and verify the result for $x = 2$ and $y = 1$:
   - (i) $\frac{1}{4}(2x^2 - 10y^2)(2x^2 + 10y^2)$
   - (ii) $(x^2 + y^2)(-2x^2 - 2y^2)$
   - (iii) $(x^2 - 5)(x + 5) + 5$
   - (iv) $5x^2 + (x + 7y)(3 - 2y)$
   - (v) $(x + y)(2x + y) + (x + 2y)(x - y)$
5. Multiply the following algebraic expressions :
   - (i) $(x + 2y)$ and $(2x - 9y + 7)$
   - (ii) $(2x - \frac{1}{2}y)$ and $\left(\frac{3}{4}x - 10y + 8\right)$
   - (iii) $(x^2 + y^2)$ and $(x + y + xy)$
   - (iv) $(a + b + c)$ and $(a^3 - b^3)$

6. Find the product :
   (i) $(1.5x - 4y)(1.5x + 4y + 3)$
   (ii) $(m^2 + n^2 + p^2)(p^2 - n^2)$
7. Simplify.
   (i) $(x + y)(x^2 - xy + y^2)$
   (ii) $x^2 + (3x - y)(3x + y + y^2)$
   (iii) $x(x + y^2 + z) + y^2(x + y + z) - z(x + y^2)$
   Verify your results in the above problem for $x = 1, y = 1$ and $z = 2$.

7.6 Standard Identities

Two simple binomials are $a + b$ and $a - b$. Multiplying each of them with itself and with the other, produces the following three products :

1. $(a + b)(a + b)$ or $(a + b)^2$
2. $(a - b)(a - b)$ or $(a - b)^2$
3. $(a + b)(a - b)$

These products occur quite often in algebra. We can easily obtain expressions for these products that are more useful by simply multiplying them out.

I. $(a + b)^2$

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a(a + b) + b(a + b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + (ab + ab) + b^2 && [\because\ ab = ba]\\ &= a^2 + 2ab + b^2 && \text{[Adding like terms]}\end{aligned}$$

Thus, $(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^2$ is often called a *binomial square* or the *square of a binomial*. Put in words,

The square of a binomial is the square of the first term plus the square of the second term plus twice the product of the two terms.

Thus, $(a + b)^2 = a^2 + b^2 + 2ab$,

which is the same as I above.

Remark : Observe that $(a + b)^2 \neq a^2 + b^2$.

One interesting fact about the relation

$$(a + b)^2 = a^2 + 2ab + b^2$$

is that this relation is true for every value of a and every value of b.

For example, taking $a = 2$, $b = 1$,

$$\text{LHS} = (a+b)^2 = (2+1)^2 = 3^2 = 9$$
$$\text{RHS} = a^2 + 2ab + b^2 = 2^2 + 2 \times 2 \times 1 + 1^2$$
$$= 4 + 4 + 1 = 9$$

Thus, LHS = RHS.

Try different pairs of values of a and b and you will find LHS = RHS This is an interesting situation because not all relations in literals have this property. For example, the relation

$$a + 3 = 2a + 1$$

is true only for one value 2 of a. For $a = 1$, this relation is not true. [Try other values of a different from 2 to convince yourself].

As another example, the relation

$$b^2 = ab + b$$

is true if $a = 1$ and $b = 2$, but not if $a = 2$ and $b = 1$

If a relation is true for *all values* of the literals that occur in this relation, we call it an *identity*. Thus, the following relation is an identity :

Identity I .

$$\boxed{\begin{array}{c}(a+b)^2 = a^2 + 2ab + b^2 \\ or \\ (a+b)^2 = a^2 + b^2 + 2ab\end{array}}$$

Activity 1 : From a piece of cardboard or an old greeting card, cut out

(i) One square piece of side 6cm.

(ii) Two rectangular pieces of sides 6cm and 4cm each.

(iii) One square piece of side 4cm.

Call these pieces A, B, C and D. The respective areas of these pieces are (6 × 6) cm^2, (6 × 4) cm^2, (6 × 4) cm^2 and (4 × 4) cm^2. Thus, the total area covered by these pieces is :

(6 × 6) cm^2 + (6 × 4) cm^2 + (6 × 4) cm^2 + (4 × 4) cm^2

or $[6^2 + 2(6 \times 4) + 4^2]$ cm^2 (1)

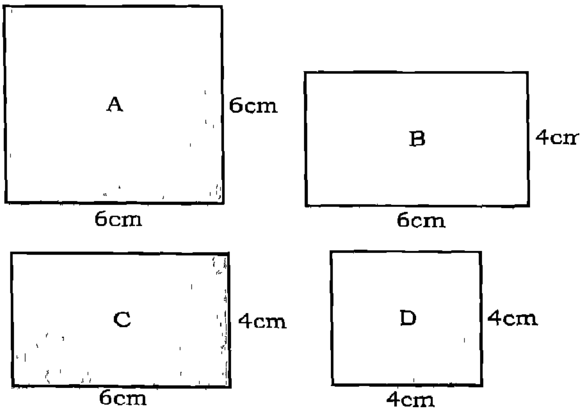


Fig 7.5

Now, can you arrange these pieces to form a square shape ? One such arrangement is shown below:

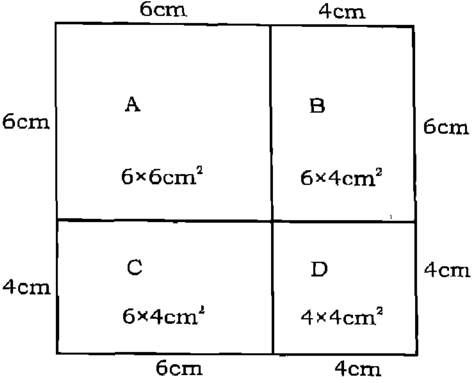


Fig 7.6

What is the side of this square piece? Obviously, its side is (6 + 4) cm What is its area ? The area is $(6 + 4) \times (6 + 4)$ cm^2 or $(6 + 4)^2$cm^2. This area is also equal to the total area of its pieces A, B, C and D. Hence,

$$(6 + 4)^2 = 6^2 + 2 \times 6 \times 4 + 4^2$$

This verifies Identity I for $a = 6$ and $b = 4$.

We may reverse the process. Instead of starting with the pieces, we may begin with the whole and cut it into pieces. Take a square piece of cardboard or old greeting card with side $(a + b)$ cm. Divide it into four pieces as shown in Fig.7.7. Cut along the dotted lines to separate out the pieces.

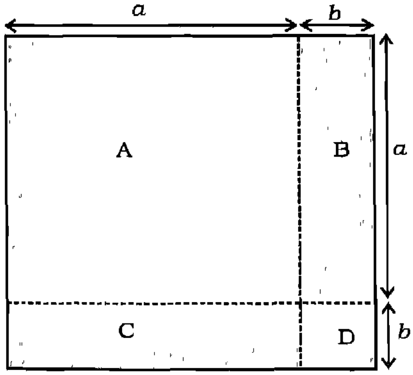


Fig. 7.7

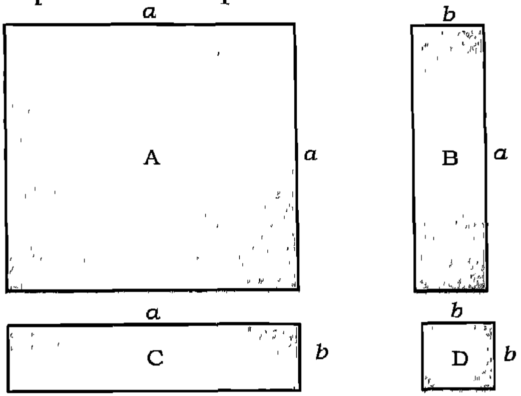


Fig. 7.8

Now the area of the original square is $(a + b)^2$. The areas of the pieces A, B, C and D respectively are a^2, ab, ab and b^2. Since the total area of the pieces must be the same as the area of the original square, we must have

$$(a + b)^2 = a^2 + ab + ab + b^2$$

or

$$(a + b)^2 = a^2 + 2ab + b^2$$

This verifies Identity I in general.

Example 9 : Evaluate :

(i) 203^2 and (ii) $(2x + 3y)^2$

by using Identity I.

Solution : (i)

$$\begin{aligned} 203^2 &= (200 + 3)^2 \\ &= 200^2 + 2 \times 200 \times 3 + 3^2 \quad \text{[Identity I]} \\ &= 40000 + 1200 + 9 \\ &= 41209 \end{aligned}$$

(ii)

$$\begin{aligned} (2x + 3y)^2 &= (2x)^2 + 2 \times 2x \times 3y + (3y)^2 \quad \text{[Identity I]} \\ &= 4x^2 + 12xy + 9y^2 \end{aligned}$$

II. $(a - b)^2$

Replacing b by $-b$ in Identity I, we get

$$(a - b)^2 = a^2 + 2a(-b) + (-b)^2$$

or

$$(a - b)^2 = a^2 - 2ab + b^2 \qquad (2)$$

Clearly, relation (2) is also true for all values of a and b, because it is nothing but Identity I where b has been given the value $-b$.

We can obtain relation (2) directly as follows :

$$\begin{aligned} (a - b)^2 &= (a - b)(a - b) \\ &= a(a - b) - b(a - b) \\ &= a^2 - ab - ba + b^2 \\ &= a^2 - 2ab + b^2 \end{aligned}$$

From the point of view of application or use, we record it as

Identity II :

$$\boxed{\begin{array}{c} (a - b)^2 = a^2 - 2ab + b^2 \\ \text{or} \\ (a - b)^2 = a^2 + b^2 - 2ab \end{array}}$$

We may word it as :

The binomial difference square $(a - b)^2$ is equal to the square of the first term plus the square of the second term minus twice the product of the two terms.

i.e., $(a-b)^2 = a^2 + b^2 - 2ab$

You must remember, however, that Identities I and II are not different even though we have written them as I and II. If you write $-b$ for b in Identity II, you get Identity I as shown below :

$$\{a-(-b)\}^2 = a^2 - 2(a)(-b) + (-b)^2$$

or $$(a+b)^2 = a^2 + 2ab + b^2$$

Recall that Identity II was obtained from Identity I by replacing b by $-b$ Thus, the two identities are really the same.

Activity 2 : Here is another interesting cardboard activity. Fig 7.9 shows one 4 × 4 square piece and two 9 × 5 rectangular pieces of cardboard. Can you arrange these three pieces so as to form a figure consisting of two squares? Pieces are not to overlap. Not as easy as the rearrangement in Activity 1, but it is not all that difficult. Play with the pieces for a while, and you would come out with some such shape as shown in Fig. 7.10.

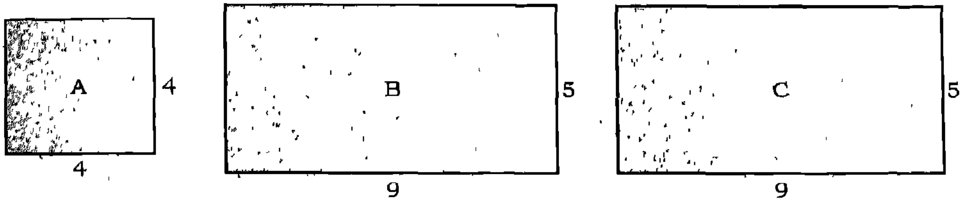


Fig. 7.9

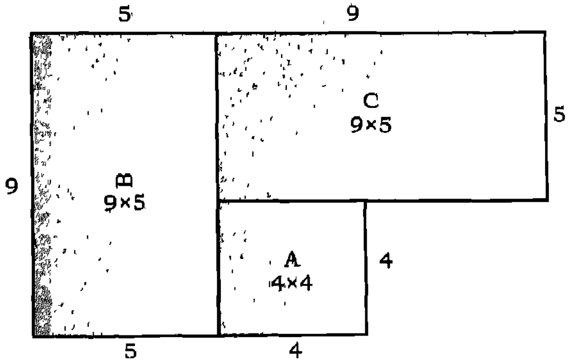


Fig. 7.10

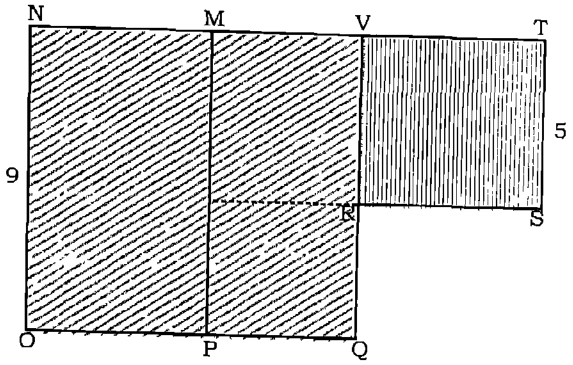


Fig 7.11

Label the shape as shown in Fig 7.11. Extend QR to meet MT in V as shown. Now

$$\begin{aligned} OQ &= OP + PQ \\ &= 5 + 4 \\ &= 9 \end{aligned}$$ [From Fig. 7.10]

Hence, OQVN is a 9 × 9 square.

Again, since MT = 9 and MV = PQ = 4,

$$\therefore VT = 9 - 4 = 5.$$

Hence, VRST is a 5 × 5 square.

Thus, Fig. 7.11 consists of two squares as desired. The total area covered by these two squares is

$$9^2 + 5^2$$

Since this figure is obtained by the three original pieces with total area

$$9 \times 5 + 9 \times 5 + 4 \times 4,$$

we have

$$9^2 + 5^2 = 2 \times 9 \times 5 + 4^2$$

or

$$9^2 + 5^2 - 2 \times 9 \times 5 = 4^2 = (9 - 5)^2$$

Thus,

$$(9 - 5)^2 = 9^2 + 5^2 - 2 \times 9 \times 5$$

But this is an example of Identity II

We can reverse this process. Let us start with two squares of sides a and b as shown in Fig. 7.12.

Draw a line PQ in the bigger square to divide it into two rectangles of dimensions $a \times b$ and $a \times (a - b)$ as shown in Fig. 7.13. Extend RS to complete Fig. 7.14. We may regard Fig. 7.14 as consisting of the rectangle APQB, rectangle RMQT and square PNST. Since

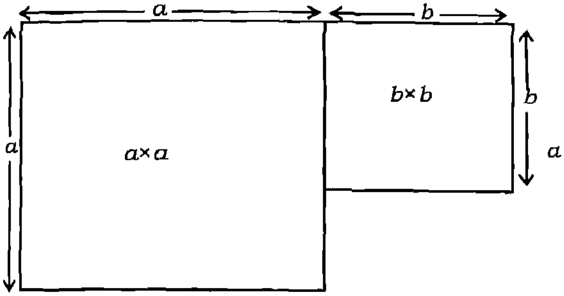


Fig. 7.12

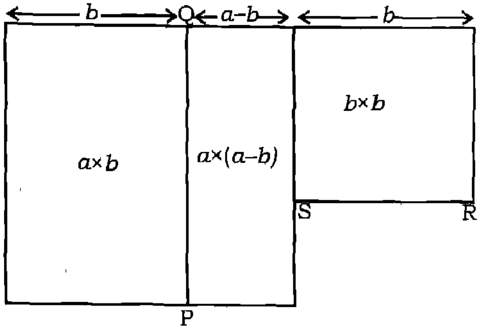


Fig. 7.13

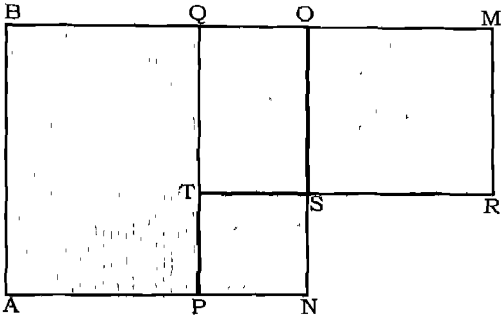


Fig 7.14

$$QM = (a - b) + b = a$$

and $NS = NO - SO = AB - RM = a - b$, therefore,

$a^2 + b^2$ = area of Fig. 7.12

= area of Fig. 7.14

= area of rectangle APQB + area of rectangle RMQT + area of square PNST

$= ab + ab + (a - b) \times (a - b)$

or $a^2 + b^2 = 2ab + (a - b)^2$

or $(a - b)^2 = a^2 - 2ab + b^2$

This verifies Identity II geometrically.

Example 10 : Evaluate (i) 4.9^2 and (ii) $(3p - 5q)^2$ by using Identity II.

Solution : (i) $4.9^2 = (5.0 - 0.1)^2$

$= (5.0)^2 - 2 \times (5.0) \times (0.1) + (0.1)^2$ [Identity II]

$= 25.0 - 1.0 + .01$

$= 24.01$

(ii) $(3p - 5q)^2 = (3p)^2 - 2 \times (3p) \times 5q + (5q)^2$ [Identity II]

$= 9p^2 - 30pq + 25q^2$

III. $(a + b)(a - b)$

Another identity, more interesting than I and II, is the following :

Identity III : $(a + b)(a - b) = a^2 - b^2$

We can easily obtain this identity by multiplication.

$$\begin{aligned}(a + b)(a - b) &= a(a - b) + b(a - b)\\ &= a^2 - ab + ba - b^2\\ &= a^2 - b^2 \qquad [\because ab = ba]\end{aligned}$$

Identity III is simpler than the first two identities as it has no product term. You may word it as

The product $(a + b)(a - b)$ of the binomial sum and difference is equal to the square of the first term minus the square of the second term.

In practice, however, we remember this identity as

Identity III : $\boxed{a^2 - b^2 = (a + b)(a - b)}$

Activity 3 : There are people who believe that many algebraic calculations may be understood by geometry in a much easier way. As we have already seen, the identities

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad \text{(I)}$$

$$\left.\begin{aligned}\text{and}\quad (a-b)^2 &= a^2 - 2ab + b^2\\ \text{or}\quad a^2 + b^2 &= (a-b)^2 + 2ab\end{aligned}\right\} \qquad \text{(II)}$$

can be viewed as follows without any explanations :

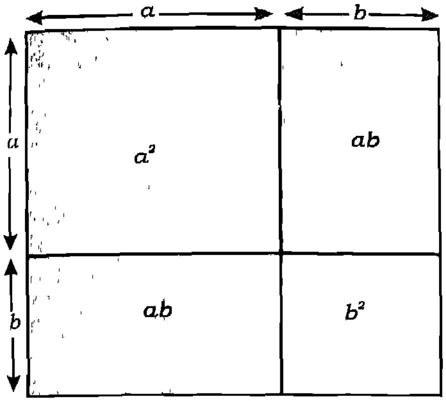


Fig. 7.15 (*Identity I*)

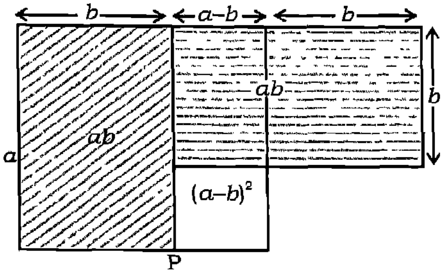


Fig. 7.16 (*Identity II*)

Let us now consider the third identity. Here is a cutting and moving puzzle, harder than the first but easier than the second :

By cutting and shifting (no overlapping), can you convert Fig. 7.17 into Fig. 7.18 or conversely?

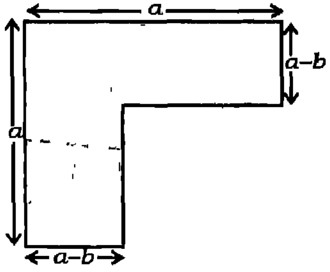


Fig. 7.17

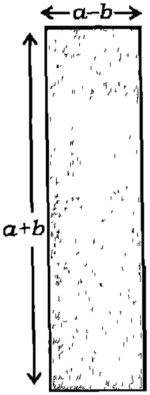


Fig. 7.18

Surely you have seen the connection of these two figures with Identity III, i.e.,

$$(a + b)(a - b) = a^2 - b^2$$

Fig. 7.18 clearly represents $(a + b)(a - b)$. In Fig. 7.17, the eaten up corner at the right bottom is a $b \times b$ piece. Thus, Fig. 7.17 represents $a^2 - b^2$. If you can solve the above puzzle, then it would be clear that both represent the same area so that

$$a^2 - b^2 = (a + b)(a - b),$$

as shown earlier, algebraically.

Now for the solution of the puzzle, one easy way to get the shape in Fig. 7.18 from that in Fig. 7.17 is shown below :

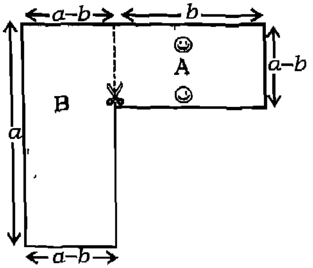


Fig 7.19 (Cut portion A)

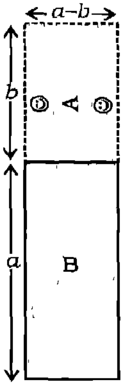


Fig. 7.20 (Put A on top of B)

To get the shape in 7.17 from that in 7.18 is clear now. Chop off a $b \times (a - b)$ rectangle from the top (Fig. 7.21) and adjoin it to the top right of the remaining portion as shown (Fig. 7.22). Thus, the third identity is also verified geometrically

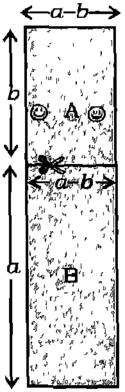


Fig 7.21 (Chop off A)

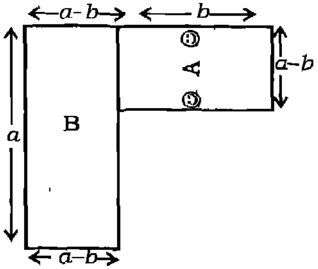


Fig. 7.22 (Adjoin A to B)

Activity 4 : We may verify Identity III in the following manner also. Take a square cardboard ABCD of side a (Fig. 7.23). Take a point P on AD and R on DC so that

$$PD = DR = b$$

Then
$$AP = RC = a - b$$

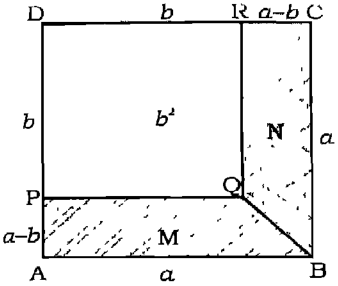


Fig. 7.23

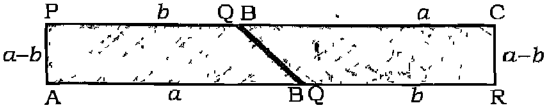


Fig. 7.24

Complete the square PQRD. Then shaded area in Fig. 7.23 is $a^2 - b^2$. Join BQ. Cut out the pieces ABQP (M) and BCRQ (N). Join these pieces as in Fig. 7 24, vertex Q of piece N lying on vertex B of piece M and vertex B of piece N lying on vertex Q of piece M. Now area of rectangle ARCP = $(a + b)(a - b)$.

Since ARCP is formed of the pieces M and N, and total area of these pieces is $a^2 - b^2$, we verify again that

$$a^2 - b^2 = (a + b)(a - b).$$

Example 11 : Evaluate (i) $981^2 - 19^2$ and (ii) 189×211 by using Identity III.

Solution : (i) $981^2 - 19^2 = (981 + 19)(981 - 19)$ [Identity III]
$= (1000)(962)$
$= 962000$

(ii) $189 \times 211 = (200 - 11)(200 + 11)$
$= 200^2 - 11^2$ [Identity III]
$= 40000 - 121$
$= 39879$

EXERCISE 7.3

1. Use a suitable identity to find the product :

(i) $(x + 3)(x + 3)$

(ii) $(2y + 5)(2y + 5)$

(iii) $\left(\frac{2}{5}p + 3\right)\left(\frac{2}{5}p + 3\right)$

(iv) $(1.1m + 2.1)(1.1m + 2.1)$

(v) $(3a + 4b)(3a + 4b)$

(vi) $\left(\frac{1}{2}x + \frac{3}{4}y\right)\left(\frac{1}{2}x + \frac{3}{4}y\right)$

2. Use a suitable identity to multiply :

(i) $(a - 5)$ and $(a - 5)$

(ii) $\left(2a - \frac{1}{2}\right)$ and $\left(2a - \frac{1}{2}\right)$

(iii) $\left(\frac{5}{2}x - 7\right)$ and $\left(\frac{5}{2}x - 7\right)$

(iv) $(7a - 9b)$ and $(7a - 9b)$

(v) $(x^2 - y^2)$ and $(x^2 - y^2)$

(vi) $(-ab + bc)$ and $(-ab + bc)$

3. Evaluate by using a suitable identity

(i) $(6x + 7)(6x - 7)$

(ii) $\left(\frac{1}{2}x - 1\right)\left(\frac{1}{2}x + 1\right)$

(iii) $(3a + 7b)(3a - 7b)$

(iv) $(-11x + 12y)(11x + 12y)$

(v) $(-a^2 + b^2)(a^2 + b^2)$

(vi) $(2x^3 + 9y^3)(2x^3 - 9y^3)$

4. Evaluate the binomial square by using an identity and verify the result by expanding directly ;

(i) $(a - 5)^2$

(ii) $(2a + 7)^2$

(iii) $(3a^2 + 4b)^2$

(iv) $(6x^2 - 5y)^2$

(v) $(-8x^3 + 7y^2)^2$

(vi) $(4m^3 + 11n^3)^2$

5. Multiply as usual and check the result by an identity :

(i) $(6x - 8y)(6x + 8y)$

(ii) $(-3a^2 + b^3)(3a^2 + b^3)$

(iii) $\left(\frac{2}{3}m^2 + \frac{3}{8}n^2\right)\left(\frac{2}{3}m^2 - \frac{3}{8}n^2\right)$

(iv) $(1.7p^3 + 1.2q^3)(1.7p^3 - 1.2q^3)$

6. Express each of the following as a trinomial ;

(i) $(a^2 - b^2)^2$

(ii) $(a^3 + b^3)^2$

(iii) $(2x + 3y^3)^2$

(iv) $(7p^3 - 5a^2)^2$

7. Simplify and express as a monomial or a binomial :

(i) $(2x+5)^2+(2x-5)^2$

(ii) $(3p+8q)^2+(3p-8q)^2$

(iii) $(150m+11n)^2-(150m-11n)^2$

(iv) $\left(2r^2-\frac{1}{400}t^2\right)-\left(2r^2+\frac{1}{400}t^2\right)^2$

8. Express each of the following as a product of a monomial and a binomial :

(i) $(ab+bc)^2-2ab^2c$

(ii) $(m^2-n^2m)^2+2m^3n^2$

9. Express as a binomial square .

(i) $(3x+7)^2-84x$

(ii) $(89p-5q)^2+1780\,pq$

10. Evaluate each of the following by using a suitable identity :

(i) 71^2 (ii) 92^2 (iii) 103^2

(iv) 59^2 (v) 99^2 (vi) 991^2

11. Simplify by expressing as the difference of two squares :

(i) 105×95 (ii) 78×82 (iii) 297×303

12. Simplify without as such calculating the square of any number .

(i) 51^2-49^2 (ii) 132^2-122^2 (iii) 233^2-227^2

13. Find the value of a, if

(i) $8a=35^2-27^2$

(ii) $9a=76^2-67^2$

(iii) $pqa=(3p+q)^2-(3p-q)^2$

14. Cut out the following pieces of cardboard and arrange the same to form a square :

(i) A 5 × 5 square piece, a 6 × 6 square piece and two 5 × 6 rectangular pieces.

(ii) A 1 × 1 square piece, a 9 × 9 square piece and two 1 × 9 rectangular pieces.

15. Cut out pieces of the given dimensions from old greeting cards and arrange them to form two squares :

(i) A 2 × 2 square piece and two 6 × 4 rectangular pieces.
[*Hint:* Have a look at Activity 2 once again.]

(ii) A 3 × 3 square piece and two 8 × 5 rectangular pieces.

Things to Remember

1. The product of two monomials is the product of their coefficients and the literals in the two monomials, the exponent of each literal being the sum of its exponents in the given monomials
2. To multiply a monomial by a binomial, we multiply the monomial with each term of the binomial and add the products
3. To multiply two binomials, we multiply each term of one binomial with each term of the other and add the products.
4. To multiply a binomial and a trinomial, we multiply each term of the binomial with each term of the trinomial and add the products.
5. **Useful Identities :** For all values of a and b,

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

FACTORISATION OF ALGEBRAIC EXPRESSIONS

CHAPTER 8

8.1 Introduction

Recall that we can multiply several given numbers to produce a single number. For example, given 2, 3 and 5, we have the *product* $2 \times 3 \times 5 = 30$ On the other hand, given a number, we can find two or more numbers whose product is the given number. For example, given 42, we have

$$2 \times 3 \times 7 = 42$$

Here 2, 3 and 7 are called the *factors* of 42 because the *product* of 2, 3 and 7 is 42.

We have learnt how to find the product of two or more algebraic expressions. It would be natural to try to find out whether we can talk about factors of a given algebraic expression, and if yes, how to find the same! In this Chapter, we shall talk about factors of an algebraic expression We shall also discuss how to factorise, i.e., how to find the factors of a given algebraic expression.

8.2 Factors of an Algebraic Expression

Let us extend the notion of a factor from numbers to algebraic expressions. In order to find the factors of a number, we write it as a product of numbers. So, to find the factors of an algebraic expression, we shall write it as a product of algebraic expressions. Every expression in the product will be a required factor Finding factors is called *factorisation.*

Illustration 1 : $x(y + z) = x \times (y + z)$

Since the product of x and $(y + z)$ is $x(y + z)$, therefore x and $(y + z)$ are factors of $x(y + z)$.

Now, $x(y + z) = xy + xz$.

Are xy and xz factors of $x(y + z)$? No! It is the *sum* of xy and xz which is equal to $x(y + z)$, and not their *product*. Hence, xy and xz are *terms* of $x(y + z)$ and not its *factors*. So, take care and learn to distinguish between a term and a factor.

Sometimes, it is possible to write an algebraic expression as a product in more than one way. To find *all* the factors, we must consider all the products. To clarify this statement, let us begin with a simple monomial $3ab$.

Illustration 2 :

$3ab = 1 \times 3ab$. $\therefore$ 1 and $3ab$ are factors of $3ab$.
$3ab = 3 \times ab$. $\therefore$ 3 and ab are factors of $3ab$.
$3ab = 3a \times b$. $\therefore$ $3a$ and b are factors of $3ab$
$3ab = a \times 3b$. $\therefore$ a and $3b$ are factors of $3ab$.
$3ab = 3 \times a \times b$. $\therefore$ 3, a and b are factors of $3ab$.

Hence, all of

1, 3, a, b, $3a$, $3b$, ab and $3ab$ are factors of $3ab$.

Thus, *if a given algebraic expression is the product of some numbers and algebraic expressions, then these numbers and algebraic expressions are called the factors of the given algebraic expression.*

We shall now learn some methods by which we can factorise a given expression, if it is possible to do so meaningfully. Recall that prime numbers 5, 7, 11, etc. cannot be factorised meaningfully. For example, the only factors of 5 are 1 and 5. Thus, it is meaningless to factorise 5. Similarly, we also have algebraic expressions that have trivial factors only.

8.3 Common Factors of Monomials

The monomial $2xy$ has factors 1, 2, x, y, $2x$, $2y$, xy and $2xy$. The monomial $5x$ has factors 1, 5, x and $5x$. Ignoring the factor 1 that is trivially a factor of every algebraic expression, we note that x is a factor of $2xy$ as well as that of $5x$. It is *common* among the factors of both of $2xy$ and $5x$. For this reason, we say that x is a *common factor* of $2xy$ and $5x$.

Illustration 3 : The non-trivial common factors of x^2y and xy are x, y and xy.

Illustration 4 : The non-trivial common factors of $10pqr$ and $5q$ are 5, q and $5q$.

Note : From now onwards, by a *factor*, we shall mean a *non-trivial factor*, unless stated otherwise.

8.3.1 Highest Common Factor of Monomials

We have seen that two monomials may have several common factors Some of these factors may be numbers and others may contain literals. For example,

9, x, y, y^2, $9x$, $9y$, $9y^2$, xy, xy^2, $9xy$ and $9xy^2$ are the common factors of the monomials $9xy^2z$ and $18x^3y^3$. We note that, out of all these, $9xy^2$ has the following properties :

1. *Its numerical coefficient is the HCF of the numerical coefficients of the given monomials,*
   [Observe that the numerical coefficients of $9xy^2z$ and $18x^3y^3$ are 9 and 18 respectively. The HCF of 9 and 18 is 9, which is the numerical coefficient of $9xy^2$.]
2. *It contains only those literals that occur in both the given monomials. The exponent of any literal in it is the least of the exponents of this literal in the given monomials.*
   [The given monomials have x and y as the common literals Only these occur in $9xy^2$ The *exponents* of x in $9xy^2z$ and $18x^3y^3$ are 1 and 3 respectively. The lesser of 1 and 3 is 1 Also, 1 is the *exponent* of x in $9xy^2$. Similarly for y.]

Because of these two properties,

(i) $9xy^2$ is a factor of both the given monomials, and
(ii) every common factor of the given monomials is a factor of $9xy^2$.

For this reason, we call $9xy^2$ as the *highest common factor* (HCF) or the greatest common factor (GCF) of the given monomials $9xy^2z$ and $18x^3y^3$

Remark : From now onwards, we shall use the term 'coefficient' for 'numerical coefficient'.

So far, we have talked about

(i) finding the factors of a given monomial, and
(ii) common factors, and HCF (GCF) of two given monomials

With the aid of these, we shall now learn to find factors of algebraic expressions. Recall that every algebraic expression is a sum of some monomials. Let us first consider a binomial which is a sum of two monomials. [Were you thinking of difference? Note that $x - y$ is the *sum* of the monomials x and $- y$.]

The following rules will be useful :

Rule 1 : $a \times b + a \times c = a \times (b + c)$

[This means that if the given binomial may be written as $a \times \ldots + a \times \ldots$, then it can be factorised as $a \times (\ldots + \ldots)$.]

Rule 2 : *A thumb-rule for finding the HCF of given monomials :*

Step 1 : Find the HCF of the coefficients of given monomials.

Step 2 : Find the least exponent of every common literal and write the literal raised to this exponent. [A constant like 7 is $7x^0y^0z^0$ .. .. .]

Step 3 : Write the HCF as the product of the HCF from Step 1 and the literals raised to the exponents from Step 2.

8.4 Factorisation by Taking out a Common Factor

We now illustrate the ideas developed so far by examples.

Example 1 : Factorise the binomial $15x^2y^3 + 12x^3y$

Solution : We shall first use the thumb-rule given above for finding the HCF of the two monomials $15x^2y^3$ and $12x^3y$ whose sum is the given binomial. With the help of the HCF, we shall find the factors.

Step 1 : The coefficients of the two monomials are 15 and 12. The HCF of 15 and 12 = $\boxed{3}$.

Step 2 : There are two literals, x and y, present in the given monomials.

(i) Exponents of x in $15x^2y^3$ and $12x^3y$ are 2 and 3 respectively. The least of 2 and 3 is 2. Hence, we write $\boxed{x^2}$.

(ii) Exponents of y are 3 and 1. So we write y^1 or $\boxed{y}$.

Step 3 : HCF of the given monomials is the product of the terms obtained in Steps 1 and 2 above.

Hence, HCF of $15x^2y^3$ and $12x^3y = 3 \times x^2 \times y$

$= 3x^2y$

We shall now re-write the terms of the given binomial so that each term is the product of two factors, one factor being the HCF $3x^2y$

$$15x^2y^3 + 12x^3y = 3x^2y \times 5y^2 + 3x^2y \times 4x$$

$$= 3x^2y\,(5y^2 + 4x) \qquad \text{[Rule 1]}$$

Hence, two factors of the given binomial $15x^2y^3 + 12x^3y$ are $3x^2y$ and $(5y^2 + 4x)$.

Remarks : 1. Once you have some practice in finding the HCF of two monomials, the above working would become very short. Find the HCF mentally and write each term as a product with one factor as the HCF, directly The detailed solution above is given so that you understand the process once for all.

2. It is not *necessary* to find out the HCF and take it out as a factor Go ahead with whatever factor appears common in both the monomials and take it out as

a factor. Then again look at the binomial factor and search for any other common factor that might be there. If you find a common factor, take it out and again examine the remaining binomial factor as shown in the next example.

Example 2 : Factorise $20a^2b^3 + 32a^3b^2$.

Solution : The two terms of the binomial are $20a^2b^3$ and $32a^3b^2$. We observe that 2 is a common factor to both these monomials. Also a^2 is a common factor in both these terms. So we take out the factor $2a^2$.

$$\therefore \quad 20a^2b^3 + 32a^3b^2 = 2a^2 \times 10b^3 + 2a^2 \times 16ab^2 \qquad (*)$$

$$\text{or} \quad 20a^2b^3 + 32a^3b^2 = 2a^2(10b^3 + 16ab^2) \qquad (1)$$

We now look at the terms of the binomial $(10b^3 + 16ab^2)$. The terms $10b^3$ and $16ab^2$ have a common factor $2b^2$ We now take this factor out.

$$\begin{aligned} 20a^2b^3 + 32a^3b^2 &= 2a^2\ (10b^3 + 16ab^2) && \text{[From (1)]} \\ &= 2a^2\ (2b^2 \times 5b + 2b^2 \times 8a) && (*) \\ &= 2a^2\ \{2b^2(5b + 8a)\} \\ &= 4a^2b^2\ (5b + 8a) \end{aligned}$$

Now $5b$ and $8a$ have no common factor. Thus, we have factorised the given expression as far as we can. Thus,

$$20a^2b^3 + 32a^3b^2 = 4a^2b^2\ (5b + 8a)$$

Remark : The steps marked (*) are important. In these steps, we have used the rule that monomials with same base are multiplied by adding exponents. .

$$\boxed{a^m \times a^n = a^{m+n}}$$

Thus, we could write :

$$a^3 = a^{2+1} = a^2 \times a^1 = a^2 \times a \text{ and } b^3 = b^2 \times b$$

Example 3 : Find the factors of $55a^2 + 22b^2$.

Solution : Here, the two terms have no common literal Hence, the HCF of $55a^2$ and $22b^2$ is merely the HCF of the coefficients 55 and 22, i.e. 11. Thus,

$$\begin{aligned} 55a^2 + 22b^2 &= 11 \times 5a^2 + 11 \times 2b^2 \\ &= 11\ (5a^2 + 2b^2) \end{aligned}$$

Therefore, 11 and $5a^2 + 2b^2$ are the required factors.

Remark : There is a worse case also. For example, $11a^2 + 13b^2$ cannot be factorised at all ! It is meaningless to write $11a^2 + 13b^2$ as $1 \times (11a^2 + 13b^2)$ The idea behind factorisation is to express the algebraic expression as a product of simpler expressions.

The road to finding factors of algebraic expressions having more than two terms is clear now. Instead of finding the HCF of two terms, we shall find the HCF of all the terms in the expression.

Example 4 : Find factors of $3ab^2 + 15a^2b^3 + 21a^3b^2$.

Solution : There are three terms $3ab^2$, $15a^2b^3$ and $21a^3b^2$ in this expression.

Step 1 : The coefficients of these terms are 3, 15 and 21. The highest common factor of these coefficients is clearly $\boxed{3}$.

Step 2 : There are two literals a and b present. The exponents of a are 1, 2 and 3. The least of these is 1. Therefore, a factor in the HCF is $\boxed{a}$ Exponents of b are 2, 3 and 2. The least of these is 2. Hence, the b-term in the HCF is $\boxed{b^2}$

Step 3 : The HCF is $3 \times a \times b^2 = 3ab^2$

Let us now re-write the given expression so that each term has $3ab^2$ as a factor.

$$3ab^2 + 15a^2b^3 + 21a^3b^2 = 3ab^2 \times 1 + 3ab^2 \times 5ab + 3ab^2 \times 7a^2$$
$$= 3ab^2\,(1 + 5ab + 7a^2)$$

Thus, $3ab^2$ and $(1 + 5ab + 7a^2)$ are factors of the given expression.

8.5 Factorisation by Regrouping Terms

Our first step towards factorising binomials and trinomials was *to take out a common factor from all the terms* of the expression Sometimes, it is not possible to do so, but by dividing the terms of the expression into groups, we can take out a common factor from each group For example, we cannot take out any common factor from

$$xy + y + 2x + 2.$$

Grouping the terms as $(xy + y) + (2x + 2)$, we can take out y common from the first and 2 from the second group. So the given expression may be written as

$$y(x + 1) + 2(x + 1)$$

Now something interesting has happened. Both groups have the factor $x + 1$ as common. So the expression can be written as

$$(x + 1) \times y + (x + 1) \times 2$$
$$= (x + 1) \times (y + 2)$$

Instead of taking out a monomial, we have taken out a binomial. This turns the expression into a product of two factors. So what have we done? We have factorised the given expression. This method of factorisation is known as *factorisation by grouping*. It is often useful in the factorisation of trinomials and expressions having four or more terms.

Example 5 : Factorise $2xy + 6x + y + 3$

Solution : We note that the first and the third terms have y as a common factor; the second and the fourth terms have 3 as a common factor. Therefore, we put the first and the third terms in one group and the remaining terms in another group. Thus,

$$\begin{aligned} 2xy + 6x + y + 3 &= (2xy + y) + (6x + 3) \\ &= y(2x + 1) + 3(2x + 1) \\ &= (2x + 1)(y + 3) \end{aligned}$$

Remark : More often than not, it is possible to group terms in many ways. In the above example, we could have proceeded as follows also :

$$\begin{aligned} 2xy + 6x + y + 3 &= (2xy + 6x) + (y + 3) \\ &= 2x \times (y + 3) + 1 \times (y + 3) \\ &= (y + 3)(2x + 1) \\ &= (2x + 1)(y + 3) \text{ as before.} \end{aligned}$$

8.6 Factorisation by Using Identities

You were told that the following identities are very useful :

I. $(a + b)^2 = a^2 + 2ab + b^2$

II. $(a - b)^2 = a^2 - 2ab + b^2$

III $(a + b)(a - b) = a^2 - b^2$

We shall now show you the utility of these identities in factorising algebraic expressions by means of examples.

Example 6 : Factorise $4a^2 - 25$.

Solution : The given expression may be written as a difference of two squares as follows :

$$\begin{aligned} 4a^2 - 25 &= (2a)^2 - 5^2 \\ &= (2a + 5)(2a - 5) \qquad \text{[Using Identity III]} \end{aligned}$$

Example 7 : Factorise $9p^2 - 16q^2$.

Solution :$$\begin{aligned} 9p^2 - 16q^2 &= (3p)^2 - (4q)^2 \\ &= (3p + 4q)(3p - 4q) \qquad \text{[By Identity III]} \end{aligned}$$

Example 8 : Factorise $49m^2 - (2n + 3l)^2$.

Solution :$$\begin{aligned} 49m^2 - (2n + 3l)^2 &= (7m)^2 - (2n + 3l)^2 \\ &= \{7m + (2n + 3l)\}\{7m - (2n + 3l)\} \\ &= (7m + 2n + 3l)(7m - 2n - 3l) \end{aligned}$$

Thus, we see that *whenever an expression is the difference of the squares of two algebraic expressions, we can use Identity III to factorise this expression.*

Another special form of expressions which gives in to easy factorisation is a trinomial which is the sum of squares of two terms with twice the product of the terms added or subtracted. In other words, expressions of the form

$$(a)^2 + (b)^2 + 2 \times a \times b \text{ or } (a)^2 + (b)^2 - 2 \times a \times b$$

Example 9 : Factorise $x^2 + 8x + 16$.

The given expression can be re-written as follows :

$$x^2 + 8x + 16 = (x)^2 + (4)^2 + 2 \times x \times 4$$

$$= (x + 4)^2 \qquad \text{[Using Identity I]}$$

$$= (x + 4)\ (x + 4)$$

Example 10 : Factorise $9m^2 + 4n^2 - 12\,mn$.

Solution : $9m^2 + 4n^2 - 12mn = (3m)^2 + (2n)^2 - 2 \times 3m \times 2n$

$$= (3m - 2n)^2 \qquad \text{[Identity II]}$$

$$= (3m - 2n)\ (3m - 2n)$$

Example 11 : Factorise $a^2 - 2ab + b^2 - c^2$

Solution : Note that the first three terms can be re-written as $(a - b)^2$ by using Identity II.

Thus, $a^2 - 2ab + b^2 - c^2 = (a - b)^2 - (c)^2$

$$= \{(a - b) + c\}\{(a - b) - c\} \qquad \text{[Identity III]}$$

$$= (a - b + c)\ (a - b - c)$$

EXERCISE 8.1

1. Find the highest common factors of the monomials .

(i) $2x^2$ and $10xy$

(ii) $21p^2q$ and $49\,pq^2$

(iii) $6a^2b^2c$ and $27abc^2$

(iv) a^3b^2 and $-7b^2$

(v) $5a^2, -25a^4$ and $100a$

(vi) $11\,abc^3$, $13a^2b^2c$ and $17abc$

(vii) $2x^3$, $4y^3$ and $6z^3$

2. Find the HCF of the terms of the expressions :

(i) $2a^2 + 10a^3 + 20a^4$

(ii) $3x^3y^2 + 9\,x^2y^3 - 12\,x^2y^2$

(iii) $-4a^5 - 16a^3b - 20a^2b^2$

(iv) $4x^2 + 20x + 40$

3. Factorise the following binomials .

(i) $7x+21$
(ii) $6p-12$
(iii) a^2+2a
(iv) $10x+5x^2$
(v) $7a^2+2a$
(vi) $3x^2y+6xy^2$
(vii) $-16m+20m^3$
(viii) $20p^2q+10apq$

4. Find the factors of the following expressions :

(i) $2x^3-6x^4-10x^2$
(ii) $-10a^3b+20b^3a+40a^3b^2$
(iii) $10a^3-15b^3+20c^3$
(iv) $a^3bc+4ab^3+41a^3$

5. Factorise each of the following ·

(i) $25a^2-b^2$
(ii) $49p^2-36$
(iii) $4a^4b^4-9p^2q^2$
(iv) a^2b^2-9
(v) $(m+2n)^2-16m^2$

6. Express as the square of an expression and hence factorise :

(i) $a^2+8a+16$
(ii) $b^2-10b+25$
(iii) $4a^2-8a+4$
(iv) $25x^2+30x+9$
(v) $49a^2+84ab+36b^2$
(vi) $121m^2-88mn+16n^2$

7. Factorise each of the following .

(i) $x(x+y)+9x+9y$
(ii) $(10xy+4x)+5y+2$
(iii) $(5x^2-20x)-8y+2xy$
(iv) $(6xy-4y)+6-9x$

8. Factorise :

(i) px^2+qx
(ii) $16x^7-48x^5$
(iii) $7x^2+21y^2$
(iv) $50x^2-72y^2$
(v) $63x^2-112y^2$
(vi) $(p-q)^2-(p+q)^2$
(vii) $2x^3+2xy^2+2xz^2$
(viii) $3a^2-9a^2b-27a^3c$

9. Factorise :

(i) $(x^2+z^2-2xz)-y^2$
(ii) $(25a^2+c^2+10ac)-49b^2$
(iii) $ap^2+bp^2+bq^2+aq^2$
(iv) $(ab+a)+b+1$

10. Find factors of the least possible degree :

(i) a^4-b^4
(ii) m^4-256
(iii) $x^4-(y+z)^4$
(iv) $x^4-(x-z)^4$

[*Hint:* Write $a^4=(a^2)^2$, $b^4=(b^2)^2$, etc]

Things to Remember

1. The number and algebraic expressions, used to form a product, are called the factors of the product
2. The highest common factor (HCF) of given monomials is obtained by multiplying the HCF of their coefficients with the product of the least powers of various literals occurring in the monomials.
3. A binomial may be factorised by taking out the HCF of the terms constituting the binomial.
4. Expressions having three or more terms may be factorised sometimes by regrouping the terms and taking out common factors from different groups of terms
5. Sometimes identities are also useful in factorising algebraic expressions

LINEAR EQUATIONS IN ONE VARIABLE

CHAPTER 9

9.1 Introduction

In the previous Class, you have learnt that an *equation* is a statement of equality that involves one or more literal numbers The literal numbers are called *unknowns* or *variables* Any value of the variable that satisfies the given equation is called a *solution* or *root* of the equation. You have also learnt some rules for solving an equation and solve some linear equations in one variable by applying these rules In all these equations, the solutions were integers. In this Chapter, we shall extend these rules for solving linear equations in one variable for which solutions could be rational numbers also. We shall also solve some simple word problems by first translating them into the form of equations and then finding the solutions of these equations.

9.2 Solving Linear Equations in One Variable

Let us recall the rules learnt in Class VI to solve an equation. These are as follows ·

The equality symbol (=) in the equation remains unchanged , if we

I. add the same number on both sides of the equation.
II. subtract the same number from both sides of the equation.
III. multiply both sides of the equation by the same non-zero number.
IV. divide both sides of the equation by the same non-zero number.

These rules are applied in such a way that we end up with only the unknown or variable appearing on one side of the equation.

Let us now solve some equations by applying the above rules :

Example 1: Solve $4x + \frac{3}{5} = 5$.

Solution : We have $4x + \frac{3}{5} = 5$

or $4x + \frac{3}{5} - \frac{3}{5} = 5 - \frac{3}{5}$ (Subtracting $\frac{3}{5}$ from both sides and using Rule II) (1)

or $4x = 5 - \frac{3}{5}$ (2)

or $5 \times 4x = 5 \times 5 - 5 \times \frac{3}{5}$ (Multiplying both sides by 5 and using Rule III)

or $20x = 25 - 3$

or $20x = 22$

or $20x \div 20 = 22 \div 20$ (Dividing both sides by 20 and using Rule IV)

or $x = \frac{22}{20} = \frac{11}{10}$

Check : For $x = \frac{11}{10}$, LHS $= 4x + \frac{3}{5} = 4 \times \frac{11}{10} + \frac{3}{5} = 5$

and RHS = 5

i.e , LHS = RHS. Hence, $x = \frac{11}{10}$ is the solution or root of the given equation.

Example 2 : Solve $\frac{x}{3} - \frac{5}{2} = 6$.

Solution : We have $\frac{x}{3} - \frac{5}{2} = 6$ (1)

or $\frac{x}{3} - \frac{5}{2} + \frac{5}{2} = 6 + \frac{5}{2}$ (Adding $\frac{5}{2}$ to both sides)

or $\frac{x}{3} = 6 + \frac{5}{2}$ (2)

or $\quad 3 \times \frac{x}{3} = 3 \times \left(6 + \frac{5}{2}\right)$ (Multiplying both sides by 3)

or $\quad x = 18 + \frac{15}{2}$

or $\quad x = \frac{51}{2}$

Thus, $x = \frac{51}{2}$ is the solution of the given equation.

Check: For $x = \frac{51}{2}$, LHS $= \frac{51}{2 \times 3} - \frac{5}{2}$

$$= \frac{17}{2} - \frac{5}{2} = \frac{12}{2} = 6$$

and $\quad$ RHS = 6

i.e., LHS = RHS. Hence, $x = \frac{51}{2}$ is the solution of the given equation

Remark : Observe the effect of subtracting $\frac{3}{5}$ from both sides of (1) of Example 1. It can be seen that $+\frac{3}{5}$ of LHS is appearing as $-\frac{3}{5}$ in the RHS [See (2)]. We say that $+\frac{3}{5}$ has been *transposed* to the other side as $-\frac{3}{5}$ Similarly, observe the effect of adding $\frac{5}{2}$ to both sides of (1) in Example 2. It can be seen that $-\frac{5}{2}$ of LHS is appearing as $+\frac{5}{2}$ in the RHS [See(2)]. We say that $-\frac{5}{2}$ has been *transposed* to the other side as $+\frac{5}{2}$. Thus, *by transposing a term we simply mean changing its sign and carrying it to the other side of the equation.*

In other words, the + sign of the term changes into the – sign on the other side and vice-versa.

Example 3 : Solve $3x - 2(2x - 5) = 2(x + 3) - 8$

Solution : Here, we observe that the variable x appears on both sides of the equation. First, we simplify the expressions on the two sides. We can simplify the two sides simultaneously. After simplification, we can apply the rules of solving the equation. Thus, we have

$3x - 2(2x - 5) = 2(x + 3) - 8$

or $3x - 4x + 10 = 2x + 6 - 8$ (Simultaneously simplifying the two sides)

or $-x + 10 = 2x - 2$

or $-x - 2x + 10 = -2$ (Transposing $2x$ to LHS)

or $-3x + 10 = -2$

or $-3x = -2 - 10$ (Transposing + 10 to RHS)

or $-3x = -12$

or $-3x \div (-3) = -12 \div (-3)$ (Dividing both sides by –3)

or $x = 4$

Thus, $x = 4$ is the required solution.

Check : For $x = 4$, LHS $= 3 \times 4 - 2(2 \times 4 - 5)$
$= 12 - 6 = 6$

and RHS $= 2(4 + 3) - 8$
$= 6$

i.e., LHS = RHS.

Hence, $x = 4$ is the required solution of the given equation.

Example 4 : Solve $\frac{6x+1}{2}+1=\frac{7x-3}{3}$.

Solution : We have $\frac{6x+1}{2}+1=\frac{7x-3}{3}$

or $\frac{6x+1+2}{2}=\frac{7x-3}{3}$ (Simplifying the expression on LHS)

or $\frac{6x+3}{2}=\frac{7x-3}{3}$

or $6 \times \frac{(6x+3)}{2} = \frac{6\times(7x-3)}{3}$ (Multiplying both sides by the LCM of 2 and 3)

or $3\ (6x + 3) = 2\ (7x - 3)$

or $18x + 9 = 14x - 6$ (Simplifying the expression on both sides)

or $18x - 14x = -6 - 9$ (Transposing $14x$ from RHS to LHS and 9 from LHS to RHS)

or $4x = -15$

or $x = -\frac{15}{4}$ (Dividing both sides by 4)

Thus, $x = -\frac{15}{4}$ is the solution of the given equation.

Check : For $x = -\frac{15}{4}$, LHS $= \frac{6\times\left(-\frac{15}{4}\right)+1}{2}+1$

$$= \frac{-\frac{45}{2}+1}{2}+1$$

$$= -\frac{43}{2\times2}+1$$

$$= -\frac{43}{4}+1$$

$$= -\frac{39}{4}$$

and RHS $= \frac{7\times\left(\frac{-15}{4}\right)-3}{3} = \frac{\frac{-105-12}{4}}{3}$

$$= -\frac{117}{12} = -\frac{39}{4}$$

i.e., LHS = RHS.

Hence, $x = -\frac{15}{4}$ is the solution of the given equation.

Example 5 : Solve $0.6x + 0.8 = 0.28x + 1.16$.

Solution : We have

$0.6x + 0.8 = 0.28x + 1.16$

or $0.6x - 0.28x + 0.8 = 1.16$ (Transposing $0.28x$ to LHS)

or $0.32x + 0.8 = 1.16$

or $0.32x = 1.16 - 0.8$ (Transposing + 0.8 to RHS)

or $0.32x = 0.36$

or $\frac{0.32x}{0.32} = \frac{0.36}{0.32}$ (Dividing both sides by 0.32)

or $x = \frac{36}{32}$

or $x = \frac{9}{8}$

Thus, $x = \frac{9}{8}$ is the required solution.

You may check the correctness of the solution as in Examples 1 to 4.

EXERCISE 9.1

Solve the following equations and check your results :

1. $\frac{x}{5} + 1 = \frac{1}{15}$

2. $5x - 3 = 3x - 5$

3. $3x = 5x - \frac{8}{5}$

4. $\frac{x-8}{3} = \frac{x-3}{5}$

5. $x + 7 - \frac{16x}{3} = 12 - \frac{7x}{2}$

6. $m - \frac{m-1}{2} = 1 - \frac{m-2}{3}$

7. $\frac{6p+1}{3} + 1 = \frac{7p-3}{2}$

8. $\frac{3t-2}{3} + \frac{2t+3}{3} = t + \frac{7}{6}$

9. $3(x-3)=5(2x+1)$

10. $15(y-4)-2(y-9)+5(y+6)=0$

11. $3(5x-7)+2(9x-11)=4(8x-7)-111$

12. $4(3w+2)-5(6w-1)=2(w-8)-6(7w-4)+4w$

13. $0.16(5x-2)=0.4x+7$

14. $0.25(4y-3)=0.5y-9$

15. $2.25(2z+8)=5z-3$

16. $x-\frac{2x}{3}+\frac{x}{2}=15$

17. $\frac{x}{2}-\frac{1}{4}=\frac{x}{3}+\frac{1}{2}$

18. $2x-3(x+1)=5x-7$.

19. $18y+3y-\frac{3}{5}=21+5y-2y$

20. $\frac{4z-3}{4}-3=\frac{5z-7}{3}-4z-1$

9.3 Application of Linear Equations to Practical Problems

Many practical problems involve relations among unknown and known numbers which can be expressed in the form of mathematical expressions In Class VI, we learnt to convert some such problems into the form of algebraic expressions and in some cases in the form of equations. After obtaining an equation corresponding to a given problem, we can solve the problem by just solving the equation. These problems are often stated in words and it is for this reason, we often refer to these problems as *word problems*. Some of the *word problems* are as follows :

(i) Sum of two numbers is 52. If one of the numbers is 10 more than the other, find the numbers.

(ii) The present age of Jacob's father is three times that of Jacob. After five years, sum of their ages would be 70 years Find their present ages.

(iii) Length of a rectangle is 8m less than twice its breadth If the perimeter of the rectangle is 56m, find its length and breadth

Solving a word problem consists of two parts. First part is the formulation of the problem and setting up the equation. In the second part, we solve the equation. This solution of the equation is the solution of the word problem. The formulation part consists of the following steps .

Step 1 : Read the problem carefully and note down what is given and what is required.

Step 2 : Denote the unknown by some letter such as x, y , z, u, v, w, etc.

Step 3 : Translate the statements of the problem step/word by step/word into mathematical statements, to the extent it is possible.

Step 4 : Look for the quantities which are equal, and write an equation by writing appropriate expressions for these quantities.

So far as the second part is concerned, you already know how to solve an equation. However, it is always advisable to check the obtained solution with the conditions given in the problem. We shall illustrate these steps through some examples

Example 6 : Sum of two numbers is 52. If one of the numbers is 10 more than the other, find the numbers.

Solution : First we formulate the problem. Here, the numbers are unknown. Let us assume that the smaller number is x.

Therefore, the other number will be $x + 10$.

Now, sum of the two numbers $= x + (x + 10) = 2x + 10$

As per our problem, this sum is given to be 52.

$$\therefore \quad 2x + 10 = 52$$

This completes the formulation part. Now, we solve this equation, namely

$$2x + 10 = 52$$

$$\text{or} \quad 2x = 52 - 10 \quad \text{(Transposing 10 to RHS)}$$

$$\text{or} \quad 2x = 42$$

$$\text{or} \quad x = \frac{42}{2} \quad \text{(Dividing both sides by 2)}$$

$$\text{or} \quad x = 21$$

From this value of x, we get the solution of the word problem as follows :

Smaller number is 21 and the other number is $21 + 10 = 31$.

Thus, the two numbers are 21 and 31.

Check : $21 + 31 = 52$, i.e., the sum of the two numbers is 52. Further, $31-21= 10$, i.e., one number is 10 more than the other.

Example 7 : The present age of Jacob's father is three times that of Jacob. After five years, sum of their ages would be 70 years. Find their present ages.

Solution : Let Jacob's present age = x years.

Therefore, his father's present age = $3x$ years.

Now, after 5 years, their ages will be

Jacob : $(x + 5)$ years

Jacob's father : $(3x + 5)$ years

$\therefore$ sum of their ages after 5 years is

$(x + 5)$ years + $(3x + 5)$ years = $(4x + 10)$ years

As per our problem, it is given to be 70 years.

Therefore, $4x + 10 = 70$

or $4x = 70 - 10$

or $4x = 60$

or $x = \dfrac{60}{4} = 15$

Thus, Jacob's present age = 15 years
and his father's present age = 3×15 years = 45 years.

Check: 45 is three times of 15, i.e., age of Jacob's father is 3 times that of Jacob. Further, after five years, Jacob's age = $(15 + 5)$ years = 20 years and age of Jacob's father = $(45 + 5)$ years = 50 years.

Sum of their ages = $(20 + 50)$ years = 70 years, which is the same as given in the problem.

Example 8 : I have some 5-rupee coins and some 2-rupee coins. The number of 2-rupee coins is 4 times that of 5-rupee coins. If I have Rs 117 in all, find the number of coins of each denomination.

Solution : Let the number of 5-rupee coins be x.

$\therefore$ Number of 2-rupee coins = $4x$

$\therefore$ Total money = Rs $(x \times 5 + 4x \times 2)$
= Rs $13x$

But this is given to be Rs 117.

Therefore, $13x = 117$

or $$x = \frac{117}{13}$$

or $$x = 9$$

Thus, number of 5-rupee coins = 9

and number of 2-rupee coins = 4 × 9= 36

Check: 36 is 4 times of 9.

Also, value of the money = Rs (9 × 5 + 36 × 2)
= Rs (45 + 72) = Rs 117

which is the same as given in the problem.

Example 9 : Length of a rectangle is 8m less than twice its breadth. If the perimeter of the rectangle is 56m, find its length and breadth.

Solution : Let the breadth of the rectangle (in m) be x (Fig. 9.1).

$\therefore$ its length = $2x - 8$ (in m)

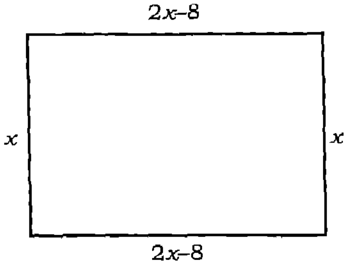


Fig 9 1

Therefore, its perimeter = $(2x - 8 + x + 2x - 8 + x)$ m
= $(6x - 16)$ m

But it is given as 56m.

$\therefore$ $$6x - 16 = 56$$

or $$6x = 56 + 16$$

or $$6x = 72$$

or $$x = \frac{72}{6}$$

or $$x = 12$$

Thus, breadth of the rectangle = 12m
and the length = $(2x - 8)$ m = $(2 \times 12 - 8)$ m = 16m
Check: Length = $(2 \times 12 - 8)$ m = 16m and perimeter = $2(16 + 12)$ m = 56m, which is the same as given in the problem.

Example 10 : Out of a swarm of bees, one-fifth part settled on a blossom of *kadamba*, and one-third on a flower of *silindhiri*, and three times the difference of those numbers flew to the bloom of a *kutaja* One bee, which remained, hovered and flew about in the air, allured at the same moment by the pleasing fragrance of a *jasmine* and *pandanus*. Tell me charming woman, the number of bees !

Solution : Let the number of bees be x.

∴ Bees settled on *kadamba* = $\frac{x}{5}$

and bees settled on *silindhiri* = $\frac{x}{3}$

∴ Difference of the numbers of bees which have flown to *kadamba* and

$$silindhiri = \frac{x}{3} - \frac{x}{5} \quad \left(\frac{x}{3} > \frac{x}{5}\right)$$

∴ Bees which flew to the *kutaja* $= 3\left(\frac{x}{3} - \frac{x}{5}\right)$

$$= x - \frac{3x}{5}$$

$$= \frac{2x}{5}$$

Total number of bees = bees on *Kadamba* + bees on *Silindhiri* + bees on *Kutaja* + 1 (left)

or $\quad x = \frac{x}{5} + \frac{x}{3} + \frac{2x}{5} + 1$

or $\quad x = \frac{3x + 5x + 6x + 15}{15}$

or $\quad 15x = 14x + 15$

or $\quad x = 15$

Thus, the number of bees was 15

Check: $15-\left[\frac{15}{5}+\frac{15}{3}+3\left(\frac{15}{3}-\frac{15}{5}\right)\right] = 15-[3+5+3(5-3)]=15-(8+6)=1$

which is the same as given in the problem

EXERCISE 9.2

1. By adding 7 to twice a number, we obtain 49 Find the number.
2. By subtracting 22 from three times a number, we obtain 68. Find the number.
3. Find a number which when multiplied by 7 and then reduced by 3 is equal to 53.
4. Sum of two numbers is 95. If one exceeds the other by 3, find the numbers.
5. Sum of three consecutive integers is 24 Find the integers.
6. Present age of Subramaniam's mother is six times Subramaniam's age. Five years hence, her age will be 20 years more than that of Subramaniam's age. Find their present ages.
7. Length of a rectangle exceeds its breadth by 4m. If the perimeter of the rectangle is 84m, find the length and breadth of the rectangle.
8. After 15 years, Sheela's age will become four times that of her present age Find her present age.
9. A sum of Rs 3000 is to be given in the form of 63 prizes. If a prize is of either Rs 100 or of Rs 25, find the number of prizes of each type.
10. A purse contains Rs 250 in notes of denominations of 10 and 50. If the number of 10-rupee notes is one more than that of 50-rupee notes, find the number of notes of each denomination
11. Length of a rectangle is 2 cm more than twice its breadth. If the perimeter of the rectangle is 28 cm, find its length and breadth
12. A number is divided into two parts such that one part is 10 more than the other If the two parts are in the ratio 5 : 3, find the number and the two parts.
13. In a Class of 35 students, the number of girls is $\frac{2}{5}$ of the boys. Find the number of boys in the Class.
14. A total of Rs 50000 is to be distributed among 200 persons as prizes. A prize is either Rs 500 or Rs 100 Find the number of each type of prizes.

15. Shanti Lal left one-fifth of his property for his son, one-fifth for his daughter and the remainder for his wife. If wife's share in the property was worth Rs 288000, find the total worth of Shanti Lal's property

16. If $\frac{1}{2}$ is subtracted from a number and the difference is multiplied by 4, the result is 5 Find the number.

17. Amarjeet has a rectangular plot of land that has been fenced with 300m long wire. Find the dimensions of the plot, if its length is twice the breadth.

18. Hamida has three boxes of different fruits. Box A weighs $2\frac{1}{2}$ kg more than Box B and Box C weighs $10\frac{1}{4}$ kg more than Box B. If the total weight of the three boxes is $48\frac{3}{4}$ kg, find the weight of Box A.

19. Sarita and Julie start walking from the same place in the opposite directions If Julie walks at a speed of $2\frac{1}{2}$ km /h and Sarita at a speed of 2km /h, in how much time will they be 18 km apart?

20. One-fourth of a herd of deer have gone to the forest. One-third of total number are grazing in a field and remaining 15 are drinking water on the bank of a river Find the total number of deer.

Things to Remember

1. Any value of the variable that satisfies the given equation is called a solution or root of the equation.

2. The equality symbol in the equation remains unchanged, if we

(i) add the same number on both sides of the equation.
(ii) subtract the same number from both sides of the equation.
(iii) multiply both sides of the equation by the same non-zero number.
(iv) divide both sides of the equation by the same non-zero number.

3. By transposing a term we mean changing its sign and taking it to the other side of the equation.

4. To solve a word problem, denote the unknown by some variable and translate the statements of the problem step / word by step/word into a mathematical statement, i.e., an equation, and then find the solution of the equation

As History Tells Us

The word 'algebra' is derived from the title of the book *Aljebar w'al almuqabalah*, written in about 825 AD by an Arab mathematician, *Mohammed ibn Al Khowarizmi* of Baghdad As a matter of fact, this book was based on the work of Indian mathematicians, who are given the credit to be the first to give a systematic study of Algebra, *Hermann Hankel*, renowned historian of mathematics, wrote,

> *"If one understands by Algebra the application of arithmetical operations to complex magnitude of all sorts, . then the learned Brahmins* (ब्राह्मण) *of Hindostan are the real inventors of Algebra."*

There is evidence that around 800 BC, Babylonians and Egyptians had a rudimentary knowledge of Algebra Babylonians were using unknowns. In Greece, *Diophantus* (250 AD) was doing some work on indeterminate equations.

However, the real story of Algebra begins in India with *Aryabhata* (born 476 AD) He was the first to appreciate the fact that similar mathematical problems could be solved all together with a general method. (Thus for example, a solution of $ax + b = 0$ gives the solution of *all* linear equations). This is the *mantra* (मत्र) of Algebra. This is what makes algebra the most useful tool of mathematics after the decimal system Thus, whereas *Diophantus* solved three equations as three problems, *Aryabhata* solved all three as a single problem.

After *Aryabhata*, *Brahmagupta* advanced his work. He had the additional tools of negative numbers, zero and fractions to facilitate his work. His method of solving a linear equation is exactly like the current method. He also solved an (indeterminate quadratic equation known as Pell's equation) equation which, centuries later, was posed as a challenge by *Fermat* (1630 AD) and solved by *Brouncker* in Brahmagupta's way. *Brahmagupta* also solved several problems of Geometry with the help of Algebra

The golden age of Algebra in India ended with *Bhaskara* (1114 AD) who realized the importance of using symbols. As evidenced from *Ahmes Papyrus* (a document written about 1550 BC) an unknown number is designated by the word *hau* (a heap), yet the credit for using single letters for unknown goes to Indians only. They used the first letters का, नी, पी (Ka, Nee, Pee), etc. of names of colours such as काला (black), नीला (blue), पीला (yellow), to denote them. The use of these letters and the method of raising them to different exponents were quite common in India.

MORE ABOUT TRIANGLES

CHAPTER

10.1 Introduction

In Class VI, we learnt about triangles and some of their important properties, namely

(i) The sum of the three angles of a triangle is 180°.

(ii) In a triangle, an exterior angle equals the sum of the two interior opposite angles.

(iii) The sum of any two sides of a triangle is greater than the third side

We also learnt classification of triangles on the basis of their sides as well as on the basis of angles.

A triangle has many other interesting properties. One of these is related to the sides and opposite angles of a triangle. It says, 'In a triangle, the angles opposite the equal sides are equal and vice-versa'. Another property relates to the right triangle. Yet another important property is the *concurrence property* of line segments associated with a triangle which we shall learn in this Chapter

All the above mentioned properties will be learnt through activities by constructing various triangles. Therefore, we begin this Chapter with construction of various triangles, when the measures of some of their sides and angles are given.

10.2 Construction of Triangles

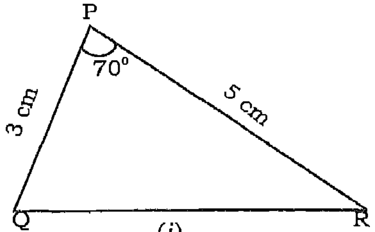
To construct a triangle with given measures, we should first draw a rough sketch and indicate the given measures. This will help us to understand the steps to be followed in the construction of a triangle

10.2.1 Construction of a triangle when two of its sides and the included angle are given (SAS Triangle Construction)

Given: The lengths of two sides of a triangle PQR where PR = 5cm, PQ = 3cm and the included angle P = 70°.

To construct: A triangle with two given sides and the included angle

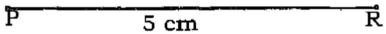
We first draw a rough sketch of ΔPQR and indicate the lengths of its sides and the measure of the included angle [Fig. 10.1(i)].



(*i*)

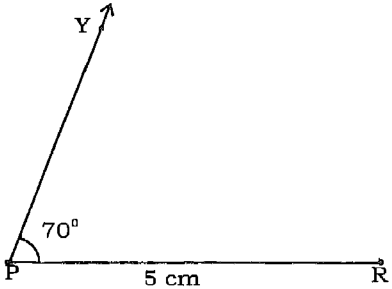
Steps of construction:

1. Draw a line segment PR of length 5cm [Fig. 10.1(ii)].



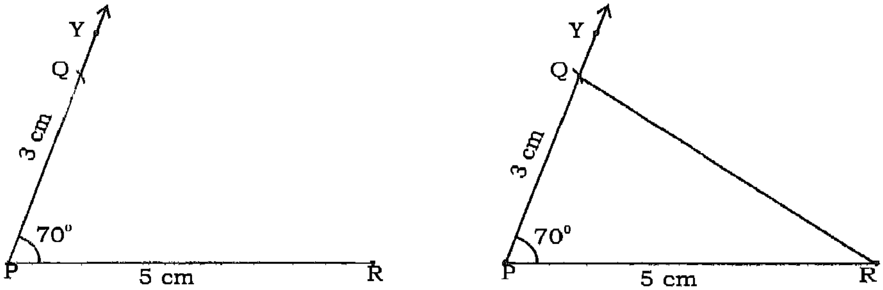
(*ii*)

2. At P, draw an angle YPR of measure 70° using a protractor [Fig. 10 1(iii)].



(*iii*)

3. From the ray PY, cut off line segment PQ of length 3cm [Fig.10.1(iv)].



(*iv*)

(*v*)

Fig. 10.1

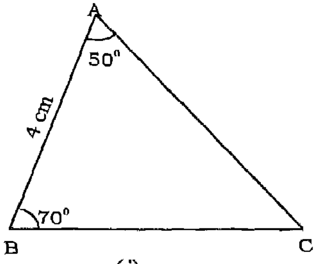
4. Join QR [Fig. 10.1(v)].

Thus, ΔPQR is the required triangle.

10.2.2 Construction of a triangle when two of its angles and the included side are given (ASA Triangle Construction)

Given: The measures of two angles of a triangle, ∠A = 50°, ∠B = 70° and the included side AB = 4 cm.

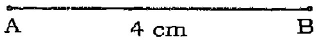
To construct: A triangle with two given angles and the included side.
We first draw a rough sketch of ΔABC and indicate the measures of two angles and the length of the included side [Fig. 10.2(i)].



(*i*)

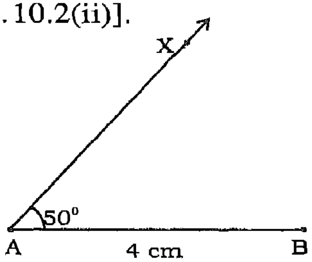
Steps of construction:

1. Draw a line segment AB of length 4 cm [Fig. 10.2(ii)].



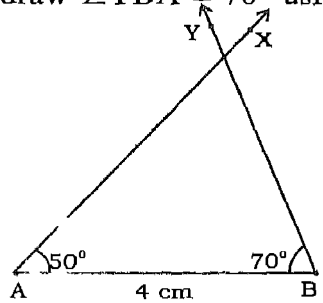
(*ii*)

2. At A, draw ∠XAB = 50° using a protractor [Fig. 10.2(iii)].



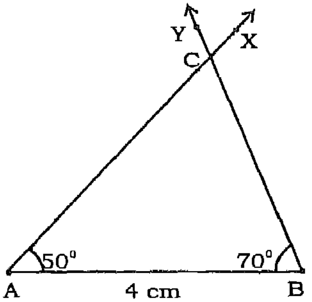
(*iii*)

3. At B, draw ∠YBA = 70° using a protractor [Fig. 10.2(iv)].



(*iv*)

4. Let AX and BY intersect at C [Fig. 10.2(v)].
Thus, ΔABC is the required triangle.



(*v*)

Fig. 10.2

Example 3 : Construct a triangle ABC in which BC = 4cm, ∠B = 110° and ∠C = 70°.

Solution : Before constructing the required triangle, let us first draw a rough sketch of the triangle.

While drawing the rough sketch (Fig.10.3), we find that it is not possible to construct this triangle Can you tell why?

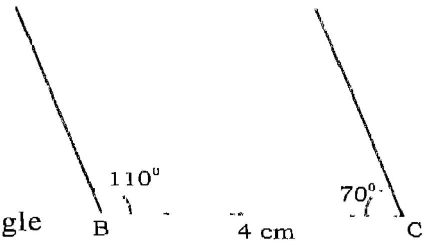


Fig 10.3

Recall the angle sum property of a triangle from Class VI It states that, in a triangle ABC,

$$\angle A + \angle B + \angle C = 180^\circ$$

Since, as given,

$$\angle B + \angle C = 180^\circ,$$
$$\angle A + \angle B + \angle C > 180^\circ.$$

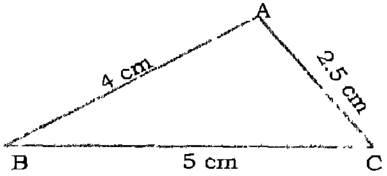
Therefore, it is not possible to construct the given triangle.

Remark : Every time you take up the construction of a triangle when two of its angles and the included side are given, i.e., under ASA condition, you are advised to check whether the given angles satisfy the angle sum property of a triangle.

Construction of a triangle when three of its sides are given (SSS Triangle Construction)

Given· The lengths of the sides of a triangle are 4cm, 5cm and 2.5cm.
To construct: A triangle with the given sides.

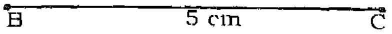
We first draw a rough sketch of ΔABC and indicate the lengths of its sides [Fig. 10.4(i)].



(*i*)

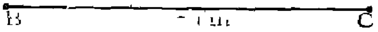
Steps of construction.

1. Draw a line segment BC = 5cm [Fig 10.4(ii)].



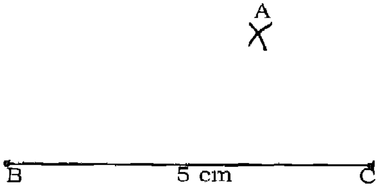
(ii)

2. With B as centre and radius 4cm (=AB), draw an arc of the circle on one side of BC [Fig. 10.4(iii)]



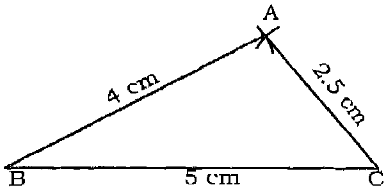
(iii)

3. With C as centre and radius 2.5cm (=AC), draw another arc intersecting the first arc at A [Fig 10.4(iv)].



(iv)

4. Join point A to B and point A to C [Fig. 10.4(v)].



(v)

Fig. 10.4

Thus, ΔABC is the required triangle.

Example 2 : Construct a triangle PQR where PQ = 2cm, QR = 3cm and PR = 6cm.

Solution : Before constructing the required triangle, let us first draw a rough sketch of the triangle.

While drawing the rough sketch (Fig.10.5), we find that it is not possible to construct this triangle. Can you tell why?

Recall the triangle inequality from Class VI. It states that the sum of any two sides of a triangle is greater than the third side.

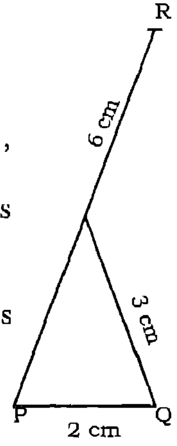


Fig. 10.5

Since in the given triangle, the sum of the lengths of the two sides PQ (2cm) and QR (3cm) is less than the length of the side PR,

i.e., $PQ + QR < PR$,

therefore the three given sides of the triangle do not satisfy the triangle in equality Hence, it is not possible to construct the given triangle.

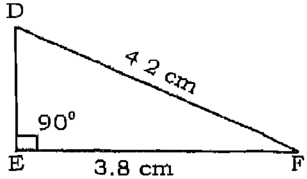
Remark : Every time you take up the construction of a triangle when all three of its sides are given, i.e., under SSS condition, you are advised to check whether the given sides satisfy the triangle in equality.

10.2.4 Construction of a right triangle when its hypotenuse and one side are given (RHS Triangle Construction)

Given: The hypotenuse DF of a right triangle DEF is 4.2cm, ∠E = 90° and EF = 3.8cm.

To construct. A right triangle, given the lengths of its hypotenuse and one of the sides.

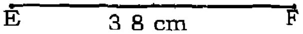
We first draw a rough sketch of ΔDEF and indicate the lengths of its hypotenuse and the given side [Fig. 10.6(i)].



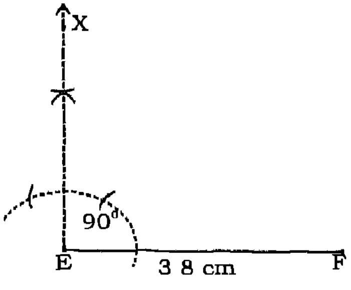
(i)

Steps of construction:

1. Draw a line segment EF of length 3.8cm [Fig. 10.6(ii)].
2. At E, draw ∠XEF of measure 90° [Fig. 10.6(iii)].

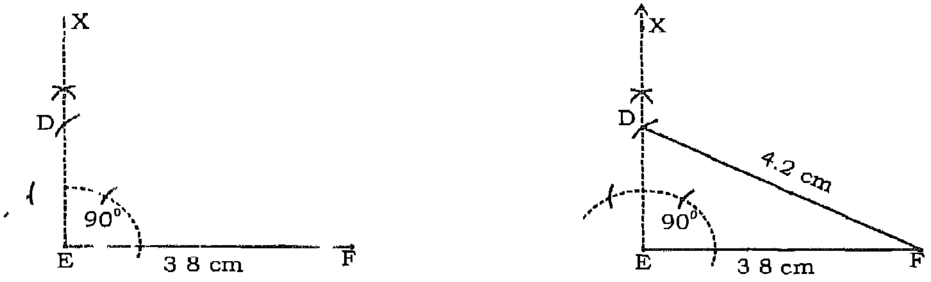


(ii)



(iii)

3. With centre F and radius 4.2cm (i.e, hypotenuse), draw an arc of the circle, to intersect ray EX at D [Fig. 10.6(iv)].



(*iv*) (*v*)

Fig. 10.6

4. Join DF [Fig. 10.6(v)]

Thus, ΔDEF is the required triangle.

EXERCISE 10.1

1. Construct a ΔABC in which $\angle B = 70^\circ$, $AB = 4.8$cm and $BC = 5.2$cm.
2. Construct an isosceles triangle XYZ in which $YZ = XZ = 4.3$cm and $\angle Z = 80^\circ$.
3. Construct a ΔDEF in which $DE = 5$cm, $DF = 4$cm and $\angle D = 50^\circ$.
4. Construct a ΔPQR in which $PQ = 4.5$cm, $QR = 4$cm and $\angle Q = 90^\circ$.
5. Construct a ΔABC in which $\angle B = 70^\circ$, $\angle C = 50^\circ$ and $BC = 5.1$cm
6. Construct a ΔDEF in which $\angle D = 100^\circ$, $\angle E = 60^\circ$ and $DE = 5\ 4$cm.
7. Construct a ΔXYZ in which $YZ = 4$cm, $\angle Y = 110^\circ$ and $\angle X = 30^\circ$. [*Hint*: Find $\angle Z$.]
8. Construct a ΔPQR in which $PQ = 5$cm, $\angle P = 40^\circ$ and $\angle R = 45^\circ$
9. Construct a triangle ABC in which $AB = 4.5$cm, $BC = 5$cm and $CA = 6$cm.
10. Construct an equilateral triangle of side 4.5cm.
11. Construct an isosceles triangle PQR in which $PQ = PR = 4.2$cm and $QR = 3\ 6$cm.
12. Construct a triangle LMN in which $LM = 5$cm, $MN = 5.6$cm and $NL = 4.2$cm.
13. Given below are the measures of some of the sides and angles of a triangle. Which of these triangles cannot be constructed? Explain, why? Construct the rest of the triangles.
    (i) $\angle A = 85^\circ$, $\angle B = 115^\circ$, $AB = 5$cm.
    (ii) $\angle Q = 30^\circ$, $\angle R = 60^\circ$, $QR = 4\ 7$cm.

(iii) $\angle A = 70^\circ$, $\angle B = 50^\circ$, AC = 3cm.
(iv) $\angle L = 95^\circ$, $\angle N = 100^\circ$, LM = 5cm.
(v) AB = 4cm, BC = 2cm, CA = 2cm.
(vi) PQ = 3.5cm, QR = 4cm, PR = 3.5cm.
(vii) XY = 3cm, YZ = 4cm, XZ = 5cm.
(viii) DE = 4.5cm, EF = 5.5cm, DF = 4cm.

14. Construct a right triangle with hypotenuse of length 5cm and one side of length 3cm.

15. Construct a right triangle ABC in which $\angle B = 90^\circ$, AB = 3cm and BC = 6.4cm.

16. Construct a right triangle PQR in which $\angle Q = 90^\circ$, PR = 6cm and QR = 4cm.

17. Construct an isosceles right triangle ABC in which $\angle C = 90^\circ$ and AC = BC = 4cm.

10.3 Properties of Isosceles Triangles

We know that in an isosceles triangle, two sides are equal. What can we say about the angles opposite to these equal sides? Are they equal? Let us see!

Activity1 : Construct a triangle ABC such that AB = AC = 3.5cm and BC = 5cm

Make a trace copy of ΔABC. Fold it so that AC falls along AB. Press to get a crease. Unfold the paper and draw the line segment AD along the crease (Fig. 10.7), meeting BC in D. Fold again so that AB falls along AC and BD falls along CD. What do we observe? We observe that $\angle C$ covers $\angle B$ exactly, or $\angle ABD = \angle ACD$.

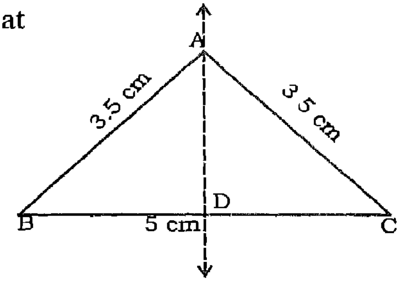


Fig. 10.7

Activity 2 : Construct a ΔABC with AB = 5cm = AC and BC = 4cm. Measure $\angle B$ and $\angle C$ and compute $\angle B - \angle C$. Repeat the above activity with two other isosceles triangles with different lengths of sides. In each case, label the triangle as ABC with AB = AC. Write the observations in the form of a table as below :

S.No.	Δ *ABC*	$\angle B$	$\angle C$	$\angle B - \angle C$
1.	AB = AC = 5cm, BC = 4cm			
2.	AB = AC = ..., BC = ...			
3.	AB = AC = ... , BC = ...			

What do we observe? We observe that in each of these cases, the difference ($\angle B - \angle C$) is either zero or so small that the same may be treated as zero.

Thus, we have

$$\angle B = \angle C$$

These activities illustrate the following proposition :

In a triangle, if two sides are equal, then the angles opposite the equal sides are also equal.

or

In an isosceles triangle, the angles opposite the equal sides are equal.

Activity 3 : Construct a triangle with $\angle B = 55° = \angle C$ and BC = 5cm (Fig. 10.8). Make a trace copy of ΔABC. Fold it so that C falls on B and the two parts of side BC cover each other exactly. Press to create a crease. What do we observe? We observe that the crease passes through A so that AC covers AB exactly. This shows that

$$AC = AB.$$

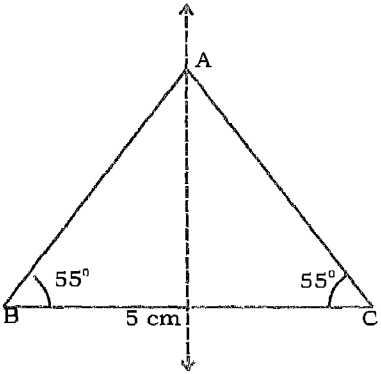


Fig 10.8

Activity 4 : Construct a ΔABC with $\angle B = \angle C = 50°$ and BC = 6cm. Measure AB and AC and compute AB – AC.

Repeat the above process with two other triangles each named ABC with different measurements but $\angle B = \angle C$. Write your observations in the form of a table as below :

S.No.	ΔABC	*AB*	*AC*	*AB – AC*
1.	BC = 6cm, $\angle B = 50° = \angle C$			
2.	BC = ..., $\angle B$ = ... = $\angle C$			
3.	BC = ..., $\angle B$ = ... = $\angle C$			

What do we observe? We observe that in each of these triangles ABC, the difference (AB – AC) is either zero or so small that the same may be treated as zero.

Thus, $AB = AC$.

These activities illustrate the following proposition :

In a triangle, if two angles are equal, then the sides opposite these angles are also equal.

Example 3 : ΔPQR is an isosceles triangle with $PQ = PR$ (Fig. 10 9). If $\angle Q = 70^\circ$, find the other two angles.

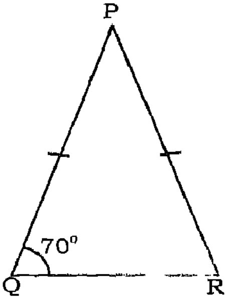


Fig. 10.9

Solution : In ΔPQR, we have

$$PQ = PR$$

$$\therefore \quad \angle Q = \angle R \qquad (1)$$

(Angles opposite to equal sides of a triangle are always equal.)

But $\angle Q = 70^\circ$ (Given)

$\therefore$ $\angle R = 70^\circ$ [By (1)]

Also, by angle sum property,

$$\angle P + \angle Q + \angle R = 180^\circ$$

Since $\angle Q + \angle R = 70^\circ + 70^\circ = 140^\circ$,

$$\therefore \quad \angle P = 180^\circ - 140^\circ$$
$$= 40^\circ$$

Thus, the required angles of the triangle are $\angle P = 40^\circ$ and $\angle R = 70^\circ$.

Example 4 : In ΔABC, $\angle A = 50^\circ$, $\angle B = 50^\circ$ and $\angle C = 80^\circ$ (Fig. 10.10). Which two sides of this triangle are equal?

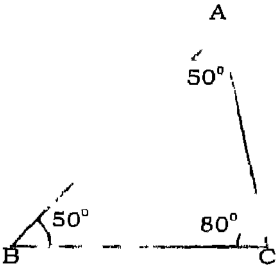


Fig. 10.10

Solution : We are given that

$$\angle A = 50^\circ = \angle B$$

Therefore, the sides opposite these equal angles must be equal.
The side opposite $\angle A$ is BC, and the side opposite $\angle B$ is AC

$\therefore$ In ΔABC, $AC = BC$.

Example 5 : ΔABC is isosceles with $AB = AC$ (Fig. 10.11). If $\angle A = 50°$, find the other two angles

Solution : In ΔABC, we have

$$AB = AC$$

$$\therefore \quad \angle C = \angle B \qquad (1)$$

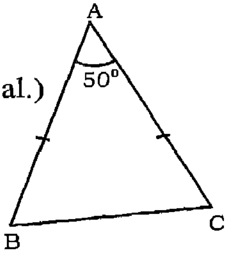


Fig 10.11

(Angles opposite equal sides of a triangle are always equal.)

Now, by angle sum property of a triangle,

$$\angle A + \angle B + \angle C = 180°$$

Since $\angle A = 50°$,

$$\therefore \quad \angle B + \angle C = 180° - 50°$$
$$= 130°$$

But $\angle B = \angle C$ [By (1) above]

$$\therefore \quad 2\angle B = 130°$$

or $\angle B = 65° = \angle C$

Example 6 : Find the value of x in Fig.10.12, where ΔPQR is isosceles with $PQ = PR$.

Solution : In Fig.10.12, we are given that

$$PQ = PR$$

$$\therefore \quad \angle R = \angle Q$$

But $\angle Q = 30°$ (Given)

$$\therefore \quad \angle R = 30°$$

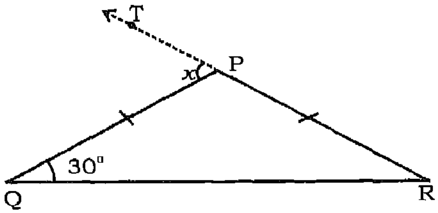


Fig 10 12

Now, by angle sum property,

$$\angle QPR + \angle Q + \angle R = 180°$$

$$\therefore \quad \angle QPR = 180° - (\angle Q + \angle R)$$
$$= 180° - 60°$$
$$= 120°$$

Now, $\angle TPQ + \angle QPR = 180°$ (Since these form a linear pair)

$$\therefore \quad \angle TPQ = 180° - 120°$$
$$= 60°$$

Thus, $x = 60°$

Example 7 : In Fig. 10.13, ΔPQR and ΔSQR are isosceles

(i) Find $\angle PQR$ and $\angle PRQ$.

(ii) Find $\angle SQR$ and $\angle SRQ$.

(iii) Find x.

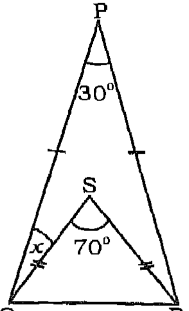


Fig. 10.13

Solution :

(i) In ΔPQR, we are given that

$$PQ = PR$$

$$\therefore \quad \angle PRQ = \angle PQR \qquad (1)$$

Now, by angle sum property,

$$\angle P + \angle PQR + \angle PRQ = 180^\circ$$

i.e., $30^\circ + \angle PQR + \angle PRQ = 180^\circ$

i.e., $\angle PQR + \angle PRQ = 180^\circ - 30^\circ = 150^\circ$

$$\therefore \quad \angle PQR = \angle PRQ = \frac{150^\circ}{2} = 75^\circ \text{ [By (1)]} \qquad (2)$$

(ii) In ΔSQR, we are given that

$$SQ = SR$$

$$\therefore \quad \angle SRQ = \angle SQR \qquad (3)$$

Now, by angle sum property,

$$\angle S + \angle SQR + \angle SRQ = 180^\circ$$

i.e., $\angle SQR + \angle SRQ = 180^\circ - \angle S$

$= 180^\circ - 70^\circ$

$= 110^\circ$

$$\therefore \quad \angle SQR = \angle SRQ = \frac{110^\circ}{2} = 55^\circ \text{ [By (3)]} \qquad (4)$$

(iii) From (2) and (4), we have

$x = \angle PQR - \angle SQR$

i.e., $x = 75^\circ - 55^\circ$

$= 20^\circ$

Thus, $x = 20^\circ$

Remark : It may be noted that in figures 10.9, 10.11, 10.12 and 10.13, the equal sides of the triangles have been shown with similar markings. Similarly, we can show equal angles of a triangle/triangles by similar markings.

EXERCISE 10.2

1. In ΔABC, $BC = CA$ Which of its two angles are equal?
2. In ΔDEF, $\angle D = \angle F$. Which of its two sides are equal?
3. ΔPQR is isosceles with $PQ = PR$. If $\angle R = 45°$, find the measures of the other two angles.
4. ΔABC is isosceles with $AB = AC$. If $\angle A = 80°$, what is the measure of $\angle C$?
5. In ΔPQR, $QP = QR$. If $\angle P = 36°$, what is the measure of $\angle Q$?
6. In ΔXYZ, $\angle X = \angle Z = 40°$(Fig. 10.14). Which of the two sides of the triangle are equal?

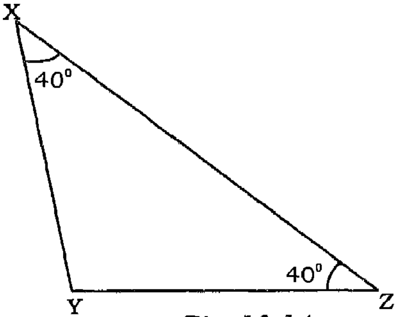


Fig 10.14

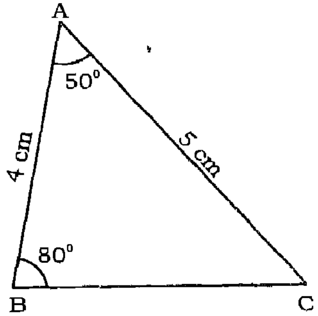


Fig. 10 15

7. In Fig. 10.15, find the length of the side BC
8. In Fig. 10.16, equal sides have been shown with similar markings. Find the values of
   (i) x (ii) y (iii) z
   Give reasons for your answer.

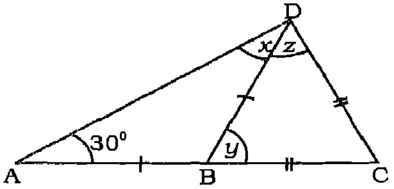


Fig. 10 16

9. In Fig. 10.17, equal sides have been shown with similar markings. Find
   (i) $\angle PRQ$ (ii) $\angle PQR$
   Give reasons for your answer.

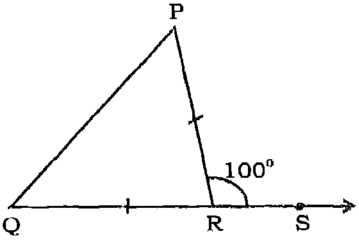


Fig. 10.17

10. In Fig. 10.18, ΔABC is isosceles with $BC = AC$. If $\angle A = 70°$, find

(i) $\angle ABC$ and $\angle ACB$.

(ii) the values of x and y.

Give reasons for your answer.

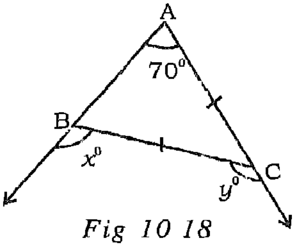


Fig 10 18

11. In Fig. 10 19, ΔPQR is isosceles with $PQ = PR$. LM is parallel to QR with L on PQ and M on PR. Give reasons for each of the following statements .

(i) $\angle Q = \angle R$.

(ii) $\angle PLM = \angle Q$.

(iii) $\angle PML = \angle R$.

(iv) $\angle PLM = \angle PML$.

(v) ΔPLM is isosceles

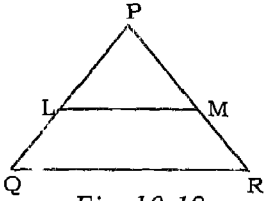


Fig. 10.19

12. In Fig. 10.20, ΔABC and ΔDBC are both isosceles with a common base BC. The equal sides have been shown with similar markings. If $\angle A = 60°$ and $\angle D = 40°$, find

(i) $\angle ABC$ and $\angle ACB$.

(ii) $\angle DBC$ and $\angle DCB$

(iii) $\angle ABD$ and $\angle ACD$. Are these equal?

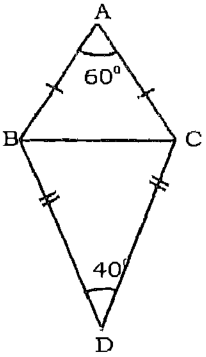


Fig 10.20

13. In Fig. 10.21, ΔPQR is isosceles with $PQ = PR$. If $\angle P$ is twice the measure of $\angle Q$, find the measures of all the angles of the triangle.

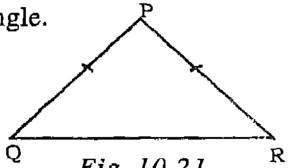


Fig. 10.21

14. In ΔPRQ (Fig.10.22), $PQ = 5cm$, $QR = 6cm$ and $PR = 4cm$.

(i) Is $\angle Q = \angle R$?

(ii) If not, say which angle is greater?

(iii) Is the greater angle opposite the longer side or the shorter side?

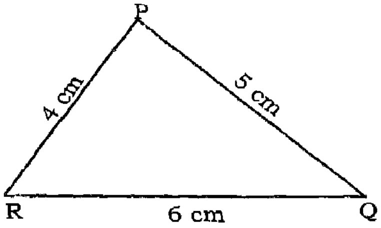


Fig. 10.22

15. In ΔABC, $BC = 5$cm, $\angle C = 40°$ and $\angle B = 50°$ (Fig. 10 23).
    - (i) Is $AB = AC$? If not, why?
    - (ii) Which of AB and AC is longer?
    - (iii) Is the longer side opposite to the greater or the smaller angle?

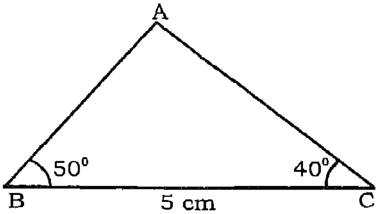


Fig 10 23

10.4 Pythagoras Theorem

Activity 5: Draw any three right triangles, each labelled as ΔPQR, $\angle Q$ in each case being a right angle (Fig. 10.24). Measure the sides p, r and the hypotenuse q in each case.

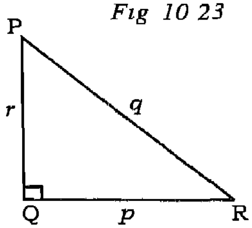


Fig. 10 24

Compute p^2, r^2 and q^2, and tabulate the result as follows :

Right Triangle	*Measurements*			*Squares*				*Differences*
	p	r	q	p^2	r^2	p^2+r^2	q^2	$q^2-(p^2+r^2)$
1.								
2.								
3.								

What do we observe? We observe that the difference $q^2 - (p^2 + r^2)$, in each case, is either zero or so small that the same may be treated as zero.

$$\therefore \quad q^2 = p^2 + r^2$$

This activity illustrates the following proposition:

In a right triangle, the square of the hypotenuse equals the sum of the squares of its sides.

In other words, if ABC is a right triangle, right angled at C (Fig. 10.25), so that AB is the hypotenuse and AC and BC are the sides of the right triangle, then

$$(AB)^2 = (AC)^2 + (BC)^2.$$

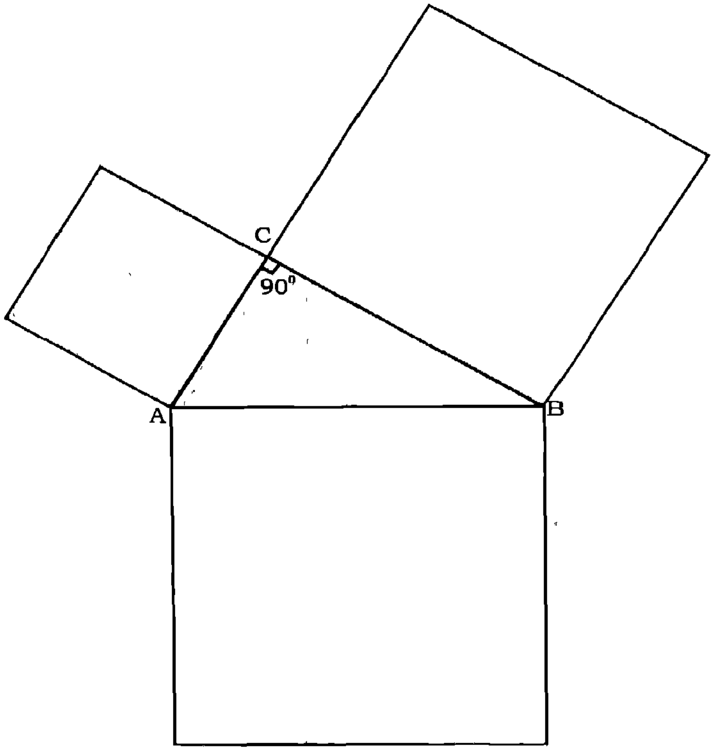


Fig. 10.25

This relation between the lengths of the sides of a right triangle is generally known as *Pythagoras Theorem*. An Indian mathematician, *Baudhayan* (about 800 BC) stated the theorem in its most general form and illustrated it by a number of numerical examples.

The *Baudhayan Theorem* may be expressed in the following words :

"The square described on the diagonal of a rectangle has an area equal to the sum of the areas of the squares described on its two sides". This means, if ABCD is a rectangle, then the area of the square drawn on the diagnonal BD is equal to the sum of areas of the squares drawn on the sides AD and AB respectively (Fig. 10.26)

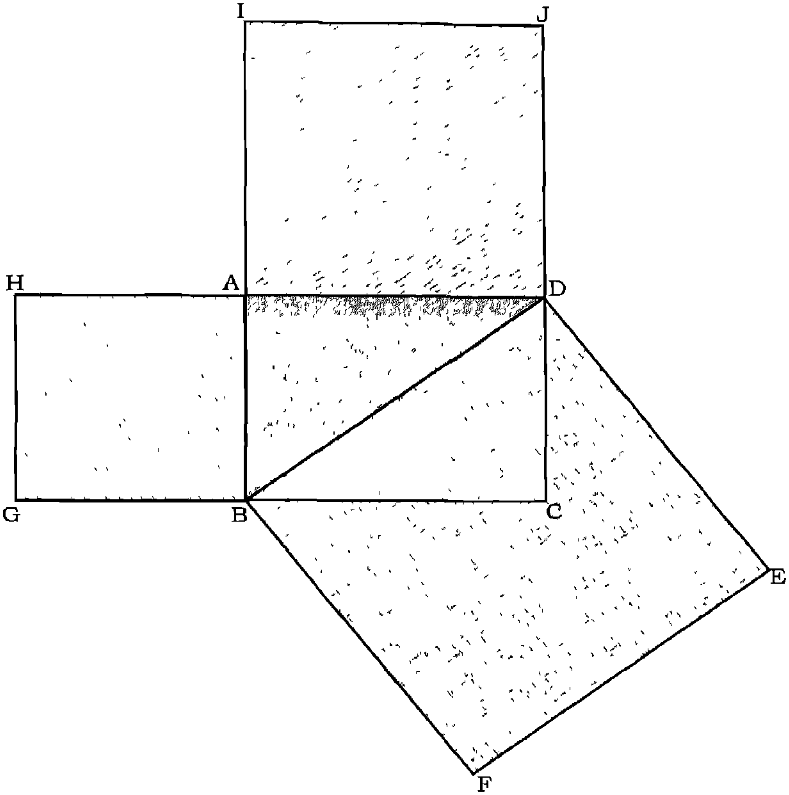


Fig. 10.26

Going back to Pythagoras Theorem, which says, if ABC is a right triangle, right angled at C, then

$$(AB)^2 = (AC)^2 + (BC)^2$$

where AB is the hypotenuse and AC and BC are the sides, we observe that

$$AB^2 > BC^2 \text{ and } AB^2 > AC^2$$

or $$AB > BC \text{ and } AB > AC.$$

Thus, we can say that, *in a right triangle, the hypotenuse is the longest side.*

10.5 Converse of Pythagoras Theorem

Activity 6 : Let ABC be a triangle (Fig. 10.27) with AB = 3cm, AC = 4cm and BC = 5cm.

Now

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$

i.e , $(AB)^2 + (AC)^2 = (BC)^2$

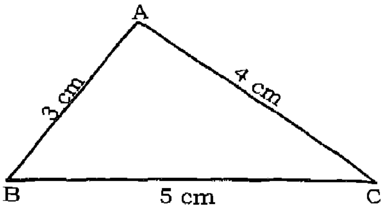


Fig 10.27

What can you say about ∠BAC? (Measure it!)

Consider another triangle DEF (Fig. 10.28) with DE = 13cm, EF = 5cm and DF = 12cm.

Now,

$$5^2 + 12^2 = 25 + 144 = 169 = 13^2$$

i.e., $(EF)^2 + (DF)^2 = (DE)^2$

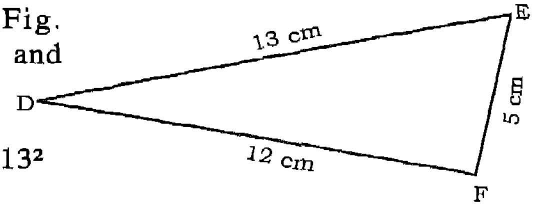


Fig. 10.28

Again, what can we say about ∠DFE?

In each case, we observe that the angle opposite the longest side, i.e., ∠BAC in the first case and ∠DFE in the second case is 90°, i.e., ΔABC is right angled at A and ΔDEF is right angled at F.

Now consider a triangle PQR with sides PQ = 4cm, QR = 5cm and PR = 6cm (Fig. 10.29).

We observe that in this triangle,

	$PQ^2 + QR^2$ is not equal to PR^2,
i.e.,	$4^2 + 5^2$ is not equal to 6^2,
or	16 + 25 is not equal to 36
or	41 is not equal to 36

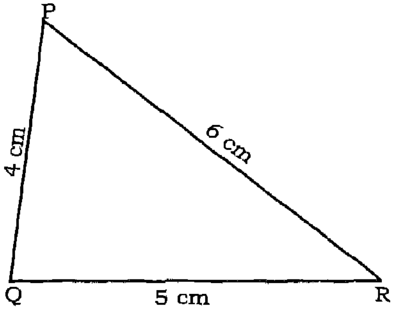


Fig. 10.29

Now measure ∠PQR, i.e., the angle opposite the longest side PR. What do you observe? We observe that ∠PQR is not equal to 90°, i.e., ΔPQR is not a right triangle.

This activity illustrates the proposition :

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, then the angle opposite the first side is a right angle.

or

We say that *the converse of Pythagoras Theorem is true.*

Three positive integers a, b and c such that $c^2 = a^2 + b^2$ are said to form a *Pythagorean triplet.*

One rule for finding Pythagorean triplets is :

For two positive integers m and n, $n > m$,
Pythagorean triplet is $a = (n^2 - m^2)$, $b = (2mn)$ and $c = (n^2 + m^2)$
Some of the Pythagorean triplets are

(3,4,5), (5,12,13), (7,24,25), (8,15,17).

If (a,b,c) is a Pythagorean triplet, then so is (ka, kb, kc) which can be obtained by multiplying or dividing all the three numbers by the same non-zero constant

Thus, for a Pythagorean triplet (3,4,5), we also have (6,8,10), (9,12,15) and so on, as Pythagorean triplets.

Example 8 : The lengths of the sides of a right triangle are 6cm and 8cm. What is the length of its hypotenuse?

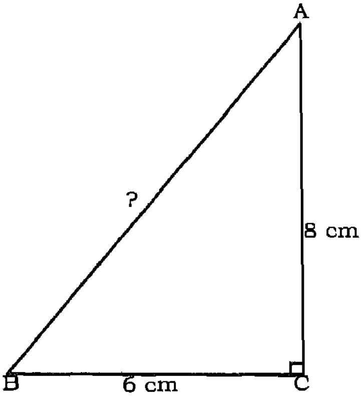


Fig 10.30

Solution : Let the lengths of the sides BC and AC of a ΔABC, right angled at C, be 6cm and 8cm respectively (Fig. 10.30).
Since ABC is a right triangle, by Pythagoras theorem,

$$(AB)^2 = (AC)^2 + (BC)^2$$

i.e., $(AB)^2 = (8)^2 + (6)^2$

$$= 64 + 36$$

$$= 100 = (10)^2$$

or AB = 10cm

Thus, the length of the hypotenuse is 10cm.

Example [illegible] A tree broke at a point but did not separate. Its top touched the ground at a distance of 5m from its base. If the point where it broke is at a height of 12m from the ground, what was the total height of the tree before it broke?

Solution : Let ABC be the tree before it broke at point B. Let the top A of the tree touch the ground at D after it broke (Fig. 10.31) so that

$$BA = BD \qquad (1)$$

Then ΔBCD is right angled at C such that BC = 12m and CD = 5m.

$\therefore$ By Pythagoras theorem,

$$\begin{aligned}(BD)^2 &= (BC)^2 + (CD)^2\\ &= (12)^2 + (5)^2\\ &= 144 + 25\\ &= 169\\ &= (13)^2\end{aligned}$$

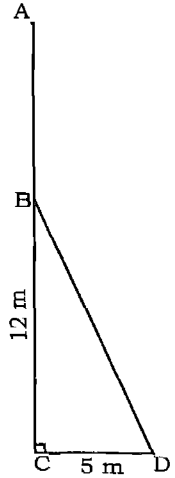


Fig 10 31

Thus, BD = 13m = AB [Using (1)]

Hence, the height of the tree before it broke was

$$CA = CB + BA = (12m + 13m) = 25m$$

Example 10 : The sides of a triangle are of lengths 6cm, 4.5cm and 7.5cm. Is this triangle a right triangle? If so, which side is the hypotenuse?

Solution : The three sides of the triangle are given to be 6cm, 4.5cm and 7.5cm The triangle will be a right triangle if it satisfies the condition :

$$(6)^2 + (4.5)^2 = (7.5)^2 \text{ (By converse of Pythagoras theorem)}$$

Now, $(6)^2 + (4.5)^2 = 36 + 20.25 = 56.25$

Also, $(7.5)^2 = 56.25$

Since the relation $(6)^2 + (4.5)^2 = (7.5)^2$ is satisfied, therefore the triangle whose sides are 6cm, 4.5cm and 7.5cm will be a right triangle.

Also, since $(7.5)^2 = (6)^2 + (4.5)^2$,

$\therefore$ the side of length 7.5cm is the hypotenuse of the triangle. In fact, the longest side of a right triangle is its hypotenuse.

EXERCISE 10.3

1. ΔABC is right angled at C. If AC = 9cm and BC = 12cm, find the length AB, using Pythagoras theorem.
2. In each of the following, the lengths of the sides of a right triangle are given. Find the square of the hypotenuse.

   (i) $a = 1.5$cm, $b = 2$cm (ii) $a = 2.5$cm, $b = 6$ cm (iii) $a = 7.5$cm, $b = 18$ cm
   (iv) $a = 14$ cm, $b = 48$ cm (v) $a = 10$ cm, $b = 24$ cm

3. The hypotenuse of a right triangle is 25cm If one of the sides is of length 24 cm, find the length of the other side
4. A ladder 17m long when set against the wall of a house just reaches a window at a height of 15m from the ground. How far is the lower end of the ladder from the base of the wall?
5. If the square of the hypotenuse of an isosceles right triangle is 200cm^2, find the length of each side.
6. Find which of the following are sides of a right triangle.

   (i) 1,1,2 (ii) 27,36,45 (iii) 14,48,50
   (iv) 15,36,39 (v) 15,10,25 (vi) 12,35,37
7. In a triangle ABC, AB = 11cm, BC = 60cm and AC = 61cm. Examine if ΔABC is a right triangle. If yes, which angle is equal to 90°?
8. Draw a triangle ABC with AC = 4cm, BC = 3cm and $\angle C = 105°$ (Fig. 10.32). Measure AB

   Is $AB^2 = (AC)^2 + (BC)^2$, If not, which of the following is true?

   $(AB)^2 > (AC)^2 + (BC)^2$

   and $(AB)^2 < (AC)^2 + (BC)^2$

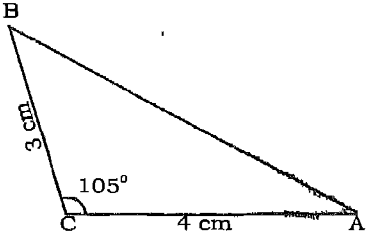
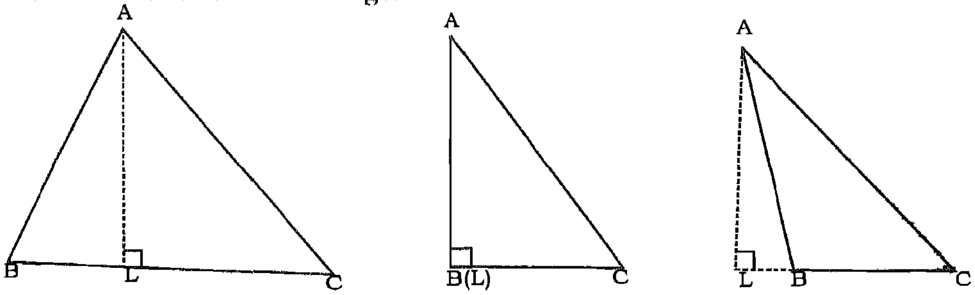


Fig. 10 32

10.6 Altitudes of a Triangle



(*i*) (*ii*) (*iii*)

Fig. 10.33

Consider three triangles [(Fig. 10.33 (i),(ii),(iii)] each labelled as ABC. In Fig. 10.33(i), ABC is an acute angled triangle, in Fig. 10.33(ii), ABC is a right angled triangle and in Fig. 10.33(iii), ABC is an obtuse angled triangle.

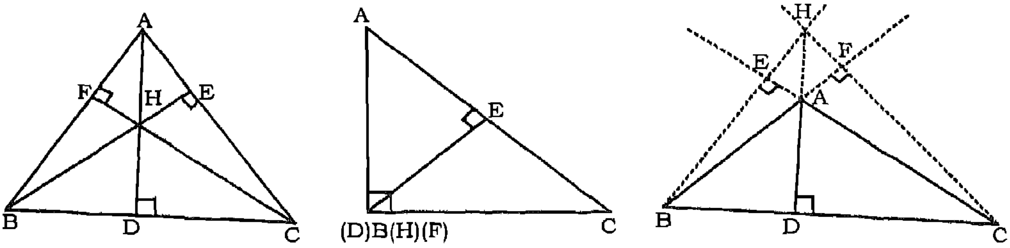
Let AL be a line segment from the vertex A, perpendicular to the line containing the side BC. Note that the position of L with respect to ΔABC is different in the three cases. If ∠B and ∠C are both acute [Fig. 10.33(i)], then L is a point on BC and is different from B and C. If ∠B is a right angle [Fig. 10.33(ii)], then L coincides with B; and if ∠B is obtuse, then L is a point on the line containing the side BC.

The line segment AL is called the *altitude* from A to BC. Thus,

A line segment from a vertex of a triangle, perpendicular to the line containing the opposite side is called an altitude of the triangle.

Every triangle has three altitudes, one from each vertex

Activity 7: Draw any ΔABC. Draw the altitudes AD and BE from the vertices A and B respectively to the line containing the opposite side [Fig. 10.34 (i),(ii),(iii)].



(*i*) (*ii*) (*iii*)

Fig. 10.34

Let these altitudes AD and BE [produced if necessary, as in Fig. 10.34(iii)] meet in H. Join CH and extend it to meet AB in F. Measure ∠CFA.

We observe that ∠CFA = 90° or CF⊥ AB. Thus, CF is the third altitude and it passes through H, the point of intersection of the other two altitudes. When we repeat this activity with more triangles, we observe the same fact.

Thus, we say that *the altitudes of a triangle are concurrent,* i.e., *they meet at a point.*

The point of concurrence of the altitudes of a triangle is called the orthocentre of the triangle.

Remarks : 1. Although an altitude of a triangle is a line segment, in the statement of the concurrence property above, we take altitude to mean a line containing the altitude (line segment).

2. To locate the orthocentre of a triangle, it is sufficient to draw any two altitudes. Further, the orthocentre of a triangle may lie inside, on or outside the triangle [Fig.10.34 (i), (ii), (iii)].

10.7 Medians of a Triangle

Consider a ΔABC. Let AD be a line segment drawn from the vertex A to the mid-point D of the opposite side [Fig.10 35]

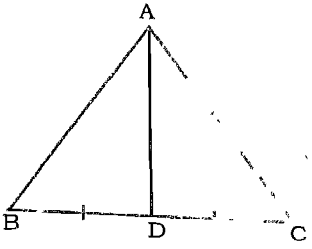


Fig. 10.35

This line segment AD is called the *median* from A to BC. Thus,

A line segment that joins a vertex of a triangle to the mid-point of the opposite side is called a median of the triangle

Since there are three vertices and three sides of a triangle, therefore *a triangle has three medians.*

Activity 8 : Draw a ΔABC. Bisect BC at D and AC at E. Join AD and BE, so that AD and BE are the two medians of the triangle ABC. Let G be the point where the medians AD and BE intersect.

Join CG and produce it to meet AB at F [Fig.10.36].

Measure AF and FB.

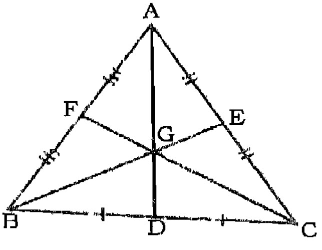


Fig. 10.36

What do we observe? We observe that AF = FB, so that F is the mid-point of AB and hence CF is the third median of the triangle ABC. When we repeat the same activity with more triangles, we observe the same fact.

Thus, we say that

the three medians of a triangle are concurrent. The point of concurrence of the medians of a triangle is called the centroid of the triangle.

Remark: To locate the centroid of a triangle, it is sufficient to draw two medians only and take the point of intersection. The centroid of a triangle always lies inside the triangle.

EXERCISE 10.4

1. Fill in the blanks :
   - (i) An altitude of a triangle is a ____________ from a vertex____________ to the opposite side.
   - (ii) The point of concurrence of the altitudes (produced, if necessary) of a triangle is called its ____________.
   - (iii) If ΔABC is an obtuse angled triangle, then its orthocentre will lie ____________ the triangle.
   - (iv) If ΔABC is right angled at C, then ____________ and ____________ are two of the altitudes of the triangle
   - (v) If H is the orthocentre of ΔABC, then BH is perpendicular to____________.
   - (vi) The medians of a triangle are____________
   - (vii) The point common to all the three medians of a triangle is called its ____________.
   - (viii) If G is the centroid of a ΔABC, then CG bisects the side____________.
   - (ix) The centroid of a triangle lies in the____________ of the triangle.
2. In Fig.10 37, ΔPQR is an isosceles triangle with PQ = PR. Draw medians QD and RE to the two equal sides.
   - (i) Verify by measurement that QD = RE
   - (ii) Is QE = RD? Give reasons.

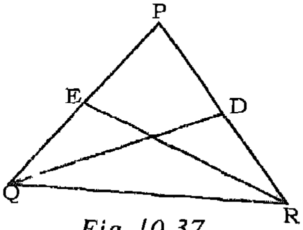


Fig. 10 37

3. ΔABC is right angled at C. Can you locate its orthocentre without drawing any altitude? If so, name it
4. Draw a triangle PQR such that $\angle Q = 110°$ Draw its altitudes PL and QM to the lines containing the opposite sides. Let the two altitudes meet at H. Join RH, meeting PQ (produced) at N

(i) Is $\angle RNQ = 90°$?

(ii) Does H lie inside or outside the triangle?

5. Draw a ΔDEF where $\angle E = 90°$. Draw medians DP and EQ from D and E respectively to the opposite sides. Let DP and EQ intersect at G. Join FG and extend it to meet DE at R. Is DR = RE? Is FR a median of the ΔDEF?
6. Show by paper-folding that the altitudes of a triangle are concurrent.
7. Draw an equilateral triangle. By paper-folding, locate its orthocentre.
8. Draw an equilateral triangle. By paper-folding, locate its centroid.

10.8 Perpendicular Bisectors of the Sides of a Triangle

Consider a ΔPQR. Let L be the mid-point of the side QR. Through L, draw $ML \perp QR$ (Fig. 10.38). Then LM is said to be the *perpendicular bisector* of the side QR of ΔPQR. Thus, *a perpendicular bisector of a side of a triangle is the line that is perpendicular to the side and bisects it.*

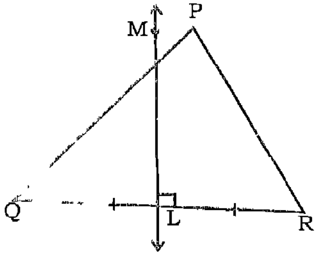
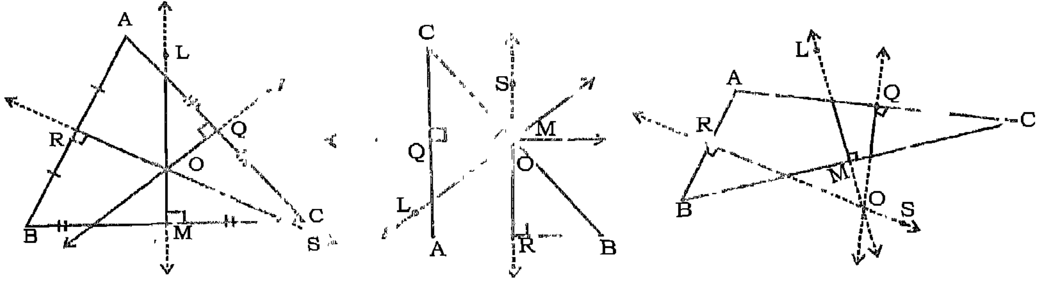


Fig. 10.38

Since a triangle has three sides, therefore it has three perpendicular bisectors, one for each side.

Activity 9 : Draw three triangles each labelled as ΔABC. In Fig. 10.39(i), ΔABC is acute angled triangle. In Fig. 10.39 (ii), ΔABC is a right triangle and in Fig. 10 39(iii), ΔABC is an obtuse angled triangle.



(i) (ii) (iii)

Fig. 10 39

Draw the perpendicular bisectors RS and ML of the sides AB and BC respectively [Fig.10.39 (i), (ii), (iii)]. Let the point of intersection of the two perpendicular bisectors be O. From O, draw OQ $\perp$ AC meeting AC in Q.

Measure AQ and QC What do we observe? We observe that AQ = QC and, therefore, OQ is the perpendicular bisector of the side AC. Thus, the point O is common to the three perpendicular bisectors of the sides of ΔABC.

When we repeat this activity with more triangles, we observe that, in each case, the perpendicular bisectors of the sides of the triangle pass through a common point. Thus, we say that *the perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrence of the perpendicular bisectors of the sides of a triangle is called the circumcentre of the triangle.*

Remarks : 1. To locate the circumcentre of a triangle, it is sufficient to draw the perpendicular bisectors of any two sides of the triangle.

2. In the case of a right triangle, its circumcentre coincides with the mid-point of the hypotenuse.

10.9 Angle Bisectors of a Triangle

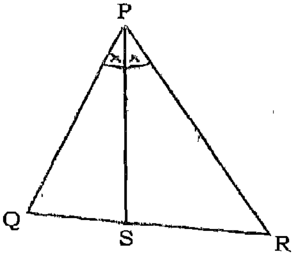


Fig. 10.40

Consider a triangle PQR. Draw the angle bisector of $\angle$QPR and let it meet QR at S (Fig.10.40). Then, PS is said to be an angle bisector of the ΔPQR. Thus, *an angle bisector of a triangle is the line segment which bisects an angle of the triangle and has its other end point on the opposite side of that angle.*

Since a triangle has three angles, therefore *it has three angle bisectors, one for each angle of the triangle.*

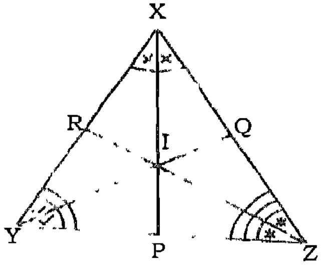


Fig. 10.41

Activity 10 : Draw a ΔXYZ. Draw XP and YQ, the bisectors of $\angle$X and $\angle$Y, respectively Let these bisectors intersect in I (Fig.10.41). Join the points I and Z and produce ZI to meet XY in R Measure $\angle$RZX and $\angle$RZY.

What do we observe? We observe that ∠RZX = ∠RZY and, therefore, the line segment ZR is the bisector of ∠Z. Thus, the third angle bisector also passes through I.

When we repeat this activity with more triangles, we observe that, in each case, the three angle bisectors of the triangle pass through a common point. Thus, we say that

the angle bisectors of a triangle are concurrent. The point of concurrence of the angle bisectors of a triangle is called its incentre.

Remark : To locate the incentre of a triangle, it is sufficient to draw any two angle bisectors.

EXERCISE 10.5

1. Fill in the blanks

   (i) The perpendicular bisectors of the sides of a triangle are________

   (ii) The circumcentre of a triangle is the point of concurrence of the________ of its sides.

   (iii) The angle bisectors of a triangle are________

   (iv) The incentre of a triangle is the point of concurrence of its ________

   (v) If I is the incentre of ΔABC, then AI bisects________.

2. Draw a triangle ABC with sides AB = 5cm, ∠B = 70° and BC = 6cm. Find its incentre.

3. Draw a triangle PQR with QR = 4.5cm, ∠R = 110° and PR = 7cm Find its circumcentre Does it lie inside the triangle PQR?

4. Draw an isosceles ΔPQR with PQ = PR. Let PS bisect ∠P. Find the circumcentre, orthocentre and centroid of ΔPQR Do they lie on PS?

5. Draw an equilateral ΔDEF. Find the incentre, circumcentre, orthocentre and centroid of this triangle. Do they coincide?

6. ΔABC is equilateral. AD, BE and CF are its medians and G is the centroid (Fig. 10.42). Trace the figure on a sheet of paper. Show by paper folding that G is also the circumcentre of the ΔABC.

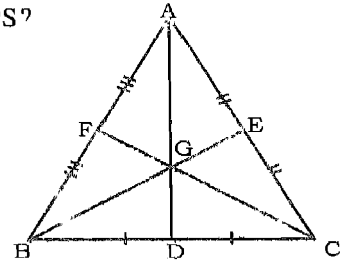


Fig. 10.42

7. Draw any triangle ABC on a sheet of paper Determine the position of the incentre of the triangle by suitable paper folding activity

Things to Remember

1. In an isosceles triangle, the angles opposite the equal sides are equal.
2. In a triangle, if two angles are equal, then the sides opposite to these angles are also equal.
3. In a right triangle, if a, b are the lengths of the sides and c that of the hypotenuse, then $c^2 = a^2 + b^2$
4. If the sides of a triangle are of lengths a, b and c, such that $c^2 = a^2 + b^2$, then the triangle is right angled and the side of length c is its hypotenuse.
5. The medians of a triangle are concurrent.
6. The altitudes of a triangle are concurrent
7. The perpendicular bisectors of the sides of a triangle are concurrent.
8. The angle bisectors of a triangle are concurrent.
9. The centroid of a triangle is the point of concurrence of its medians
10. The orthocentre of a triangle is the point of concurrence of its altitudes.
11. The circumcentre of a triangle is the point of concurrence of the perpendicular bisectors of its sides.
12. The incentre of a triangle is the point of concurrence of its angle bisectors.

CONGRUENT TRIANGLES

CHAPTER 11

11.1 Introduction

In our day-to-day life, we come across many objects with same shape and size For example, keys of the same lock, fifty paise coins, sheets of paper from the same letter pad, toys coming out of the same mould, shaving blades of the same brand, etc. Objects, which have the same shape and size are called *congruent objects*. The relation of two objects being congruent is called *congruence*.

In the present Chapter, we shall confine ourselves to the congruence relation among plane figures only. In other words, we shall learn about such figures which have the same shape and same size and lie in a plane.

11.2 Congruence of Figures

Suppose we are given two plane figures F_1 and F_2 (Fig. 11.1). How can we check whether these are congruent or not? We trace one figure, say F_1, on a trace paper and place this trace copy of F_1 over F_2. While placing trace copy of F_1 over F_2, we are permitted the following motions :

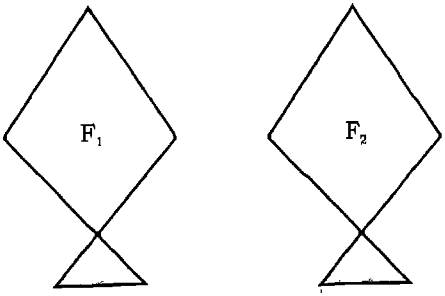


Fig 11.1

(i) Sliding or translating, i.e., moving in the plane without rotation.
(ii) Rotation about a point in the plane.
(iii) Going out of the plane and inverting upside down.

We are not permitted any other motion like bending, stretching and twisting

With the three types of motions (i) to (iii) mentioned above, we try to move trace copy of F_1 so as to cover F_2. If F_1 covers F_2 exactly, then F_1 and F_2 must be of the same shape and same size. Therefore, we can say that F_1 is congruent to F_2.

This method of comparing figures is called *method of superposition*

Thus, a plane *figure* F_1 *is said to be congruent to* F_2, *if* F_1, *when superposed over* F_2, *covers it exactly.*

The symbol '≅' is used to indicate 'is congruent to'.

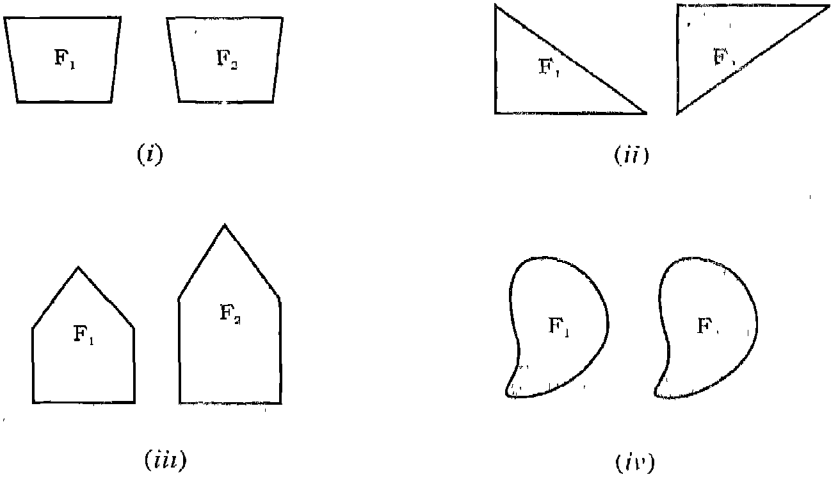
Thus, when F_1 is congruent to F_2, we write it as

$F_1 \cong F_2$ and read it as *figure* F_1 *is congruent to figure* F_2

It should be noted that if $F_1 \cong F_2$, then $F_2 \cong F_1$.

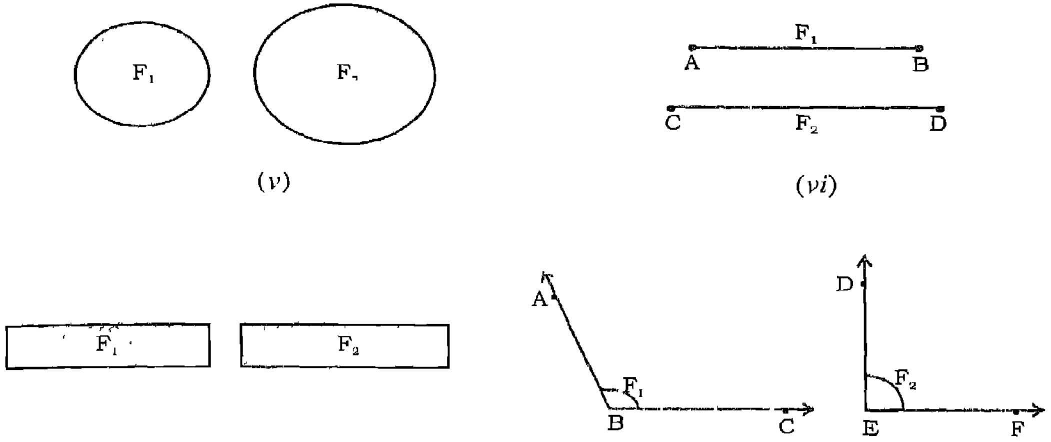
Thus, we may also say that figures F_1 and F_2 are congruent (to each other). We already know about several congruent figures from our earlier experiences. For example, when we bisect a line segment, we get two congruent line segments. Similarly, when we bisect an angle, we obtain two angles which are congruent to each other.

Example 1 : Given below are some pairs of figures (Fig 11.2). By the method of superposition, find which of them are congruent.



(i) (ii)

(iii) (iv)



(v) (vi)

(vii) (viii)

Fig. 11.2

Solution : In each case, we trace F_1 and place it over F_2. We find that

(i) $F_1 \cong F_2$ as F_1 covers F_2 exactly, i.e., F_1 and F_2 are of same shape and size.

(ii) $F_1 \cong F_2$, since by rotation, F_1 covers F_2 exactly, i.e., F_1 and F_2 are of same shape and same size.

(iii) F_1 is not congruent to F_2, since F_1 does not cover F_2 exactly. Here, F_1 and F_2 are of same shape but not of same size.

(iv) $F_1 \cong F_2$, since F_1 covers F_2 exactly, i.e., F_1 and F_2 are of same shape and size.

(v) F_1 is not congruent to F_2, since F_1 does not cover F_2 exactly. Here, F_1 and F_2 are of same shape but not of same size.

(vi) F_1 is not congruent to F_2, since the two line segments are not of same size (length).

(vii) $F_1 \cong F_2$, since F_1 covers F_2 exactly, i.e., the two rectangles are same in size.

(viii) F_1 is not congruent to F_2, since F_1 does not cover F_2 exactly. Here, the two angles are not of same size (measure).

Remarks : 1. Two line segments are congruent, if they have the same length. We often say ‘two line segments are equal’ to mean that they are congruent.

2. Two angles are congruent, if they have the same measure. We often write ∠A = ∠B to mean ∠A ≅ ∠B.

11.3 Congruence of Triangles

Suppose we are given two triangles ABC and DEF (Fig. 11.3) and we want to examine if the two triangles are congruent. We cut out ΔDEF or make a trace copy of it and using the method of superposition, check whether the two triangles cover each other exactly If ΔDEF covers the ΔABC exactly, we say that the two triangles are congruent, otherwise not

Observe that, when ΔABC covers the ΔDEF exactly (Fig. 11.4), the vertices of ΔABC coincides with the vertices of ΔDEF. Thus, there is a matching or correspondence between the vertices of the two triangles ABC and DEF But, a warning is due here, there may be more than one matching possible. The triangles may be congruent in some matchings or correspondences and not in others. For example, using ‘$\longleftrightarrow$’ to denote a matching, in Fig. 11.5, we may match the vertices of ΔABC with that of ΔDEF

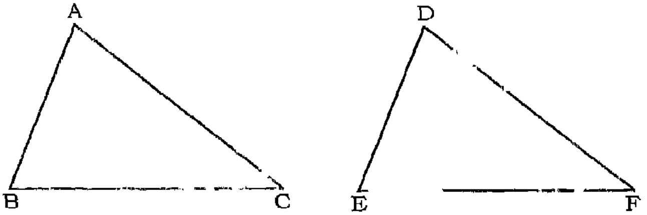


Fig. 11.3

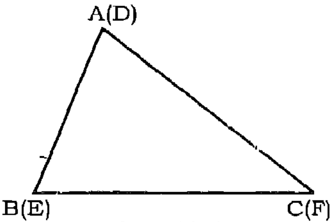


Fig 11.4

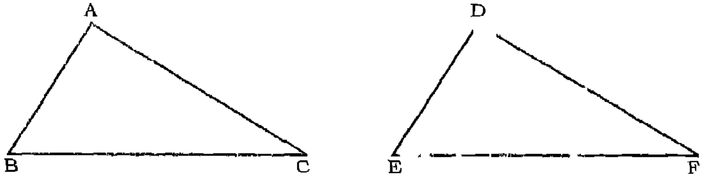


Fig 11.5

in the following manner (Fig 11.6) :

A ⟷ D, B ⟷ E, C ⟷ F

i.e., ABC ⟷ DEF.

Note that in a matching, the order of the letters is important.

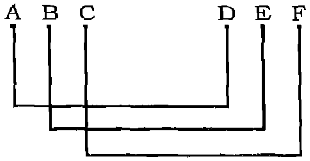


Fig. 11.6

It can be easily seen that there are six possible ways of matching the vertices of ΔABC to that of ΔDEF. These are :

ABC ⟷ DEF ABC ⟷ DFE ABC ⟷ EDF
ABC ⟷ EFD ABC ⟷ FDE ABC ⟷ FED

If in any one of these correspondences (i.e , the process of matching the vertices), the two triangles coincide, then we say that the two triangles are congruent.

In case of ΔABC and ΔDEF of Fig. 11.5, we can see that the two triangles coincide when we make the matching ABC ⟷ DEF. So we say that under the matching ABC ⟷ DEF, ΔABC is congruent to ΔDEF The other matchings do not give a congruence.

Observe that each matching between the vertices of two triangles determines a matching of their parts as well. Thus, the matching ABC ⟷ DEF leads to the matchings

(side AB)⟷(side DE),(side BC)⟷(side EF), (side AC)⟷(side DF)
∠A ⟷ ∠D, ∠B ⟷ ∠E, ∠C ⟷ ∠F.

Matching parts are also called the *corresponding parts* of the two triangles.

Thus, when we say ΔABC ≅ ΔDEF, we mean to say that the two triangles ABC and DEF are congruent under the correspondence ABC ⟷ DEF between the vertices.

Again, ΔABC ≅ ΔDEF means, if we superpose ΔDEF on ΔABC with D on A, E on B and F on C, then the two triangles cover each other exactly. It follows immediately that

AB = DE, BC = EF and AC = DF;
∠A = ∠D, ∠B = ∠E and ∠C = ∠F.

i e., the matching parts (corresponding parts) of congruent triangles are equal.

It can be shown that the converse is also true, i.e., if there is a correspondence between the vertices of two triangles so that the corresponding parts of the two triangles are equal, then the two triangles are congruent.

Example 2 : In Fig.11.7, the measures of the sides and angles of two congruent triangles have been indicated. Determine the correspondence between the two triangles which gives a congruence.

Solution : From the figures, we note that

$\angle A = \angle E$, $\angle C = \angle D$ and, therefore, $\angle B = \angle F$.

Also, BC = DF, AB = EF and AC = ED.

Thus, the correspondence ABC$\longleftrightarrow$EFD gives a congruence.

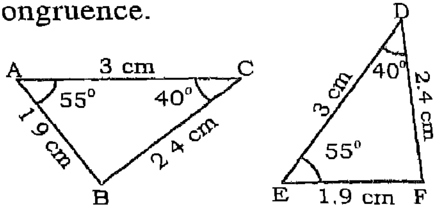


Fig. 11.7

Example 3 : In Fig.11.8, two triangles PQR and XYZ have been drawn. The measures of their sides and angles are also indicated. Find which of the following statements is true :

(i) $\Delta PQR \cong \Delta XYZ$ (ii) $\Delta PQR \cong \Delta ZYX$ (iii) $\Delta PQR \cong \Delta YXZ$

Solution : From Fig.11.8, we find that

$\angle P = \angle Z$, $\angle Q = \angle Y$ and $\angle R = \angle X$;

Also, PQ = ZY, QR = YX and PR = ZX.

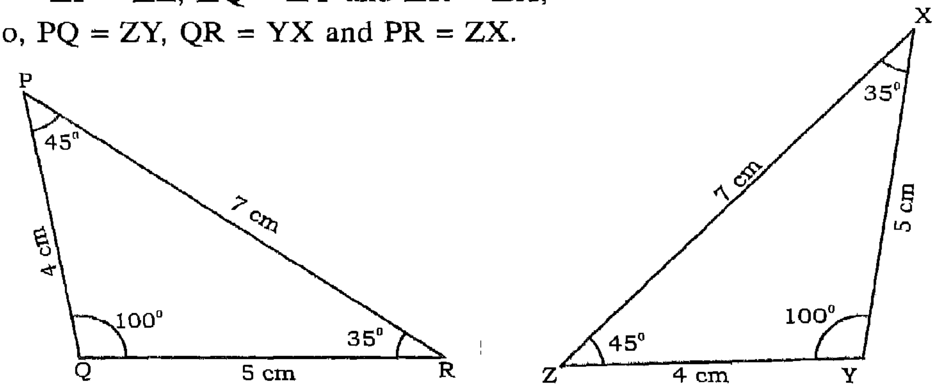


Fig. 11.8

Thus, the correspondence PQR $\longleftrightarrow$ ZYX gives a congruence. Hence, $\Delta PQR \cong \Delta ZYX$. Thus, statement (ii) is true.

11.4 Conditions for Congruence of Triangles

In the previous section, we learnt that for two triangles to be congruent, it is necessary that all three corresponding sides and all three corresponding angles of the two triangles be equal However, now we will see that all the six conditions are not necessarily required to be verified for establishing the congruence of two triangles. When some of the conditions hold good, the others are found to be true automatically.

Let us find the least possible number of conditions that are required for establishing the congruence of two triangles.

Recall that we can construct a triangle when

(i) the measures of its two sides and the included angle (SAS) are given,
(ii) the measures of its two angles and the included side are given (ASA),
(iii) the measures of its three sides are given (SSS), and
(iv) one angle is given to be 90°, and the measures of one side and the hypotenuse are given (RHS).

Let us construct two triangles with similar given measures and try to establish congruence between them.

11.4.1 The 'Side Angle Side' (SAS) Congruence Condition

Activity 1 : Construct a ΔABC with $AB = 5cm$, $BC = 6cm$ and the included $\angle B = 120^\circ$. Draw another ΔDEF with $DE = 5cm$, $EF = 6cm$ and the included $\angle E = 120^\circ$. Then, we have

$AB = DE$, $BC = EF$ and the included angles $\angle B = \angle E$ (Fig. 11.9).

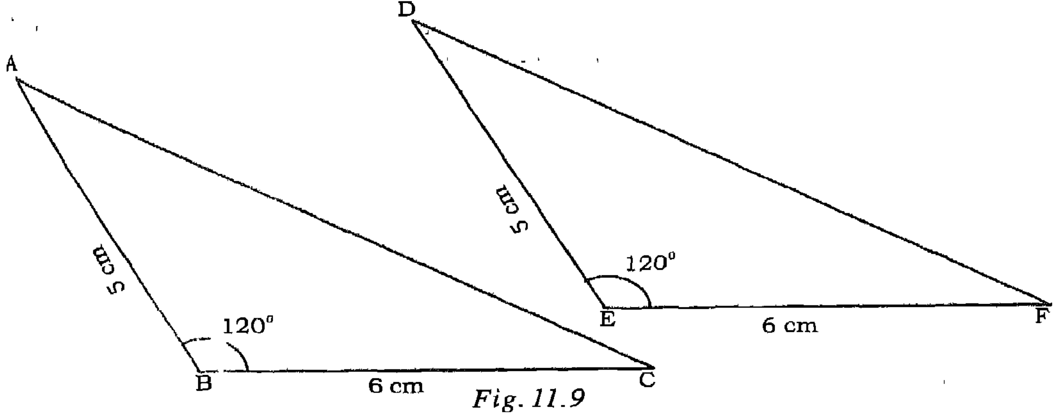


Fig. 11.9

Make a trace copy of ΔDEF and place it over ΔABC. What do we observe? We observe that with vertices D over A, E over B, and F over C, the ΔDEF exactly covers the ΔABC, i.e., under the correspondence ABC←→DEF, ΔDEF completely covers ΔABC. So we say that

$$\Delta ABC \cong \Delta DEF$$

Activity 2 : Construct a ΔABC with AB = 4 cm, AC = 5cm, and the included ∠A = 30°. Draw another ΔPQR with PQ = 4cm, PR = 5cm and the included ∠P = 30°. Then we have AB = PQ, AC = PR and the included angles ∠A = ∠P (Fig. 11.10).

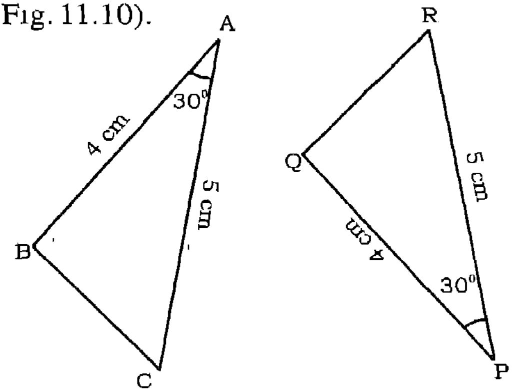


Fig. 11 10

Measure the remaining parts of the two triangles and state the observations in the form of a table, as below :

Remaining parts of ΔABC	*Corresponding parts of* ΔPQR	*Differences*
BC =	QR =	BC – QR =
∠B =	∠Q =	∠B – ∠Q =
∠C =	∠R =	∠C – ∠R =

We observe that differences (BC – QR), (∠B – ∠Q) and (∠C – ∠R) are either zero or so small that the same may be treated as zero. Thus,

BC = QR, ∠B = ∠Q and ∠C = ∠R

Therefore, ΔABC ≅ ΔPQR.

Observe that in both the activities, we took two triangles in which two sides and the included angle of one are equal to the corresponding two sides and the included angle of the other. The two triangles so formed are found to be congruent. These situations are special cases of the following proposition:

Two triangles are congruent, if two sides and the included angle of one triangle are respectively equal to the two sides and the included angle of the other.

This is called SAS *congruence condition.*

Note: The angle in SAS congruence condition should be included between the two considered sides. For example, in Fig 11 11,

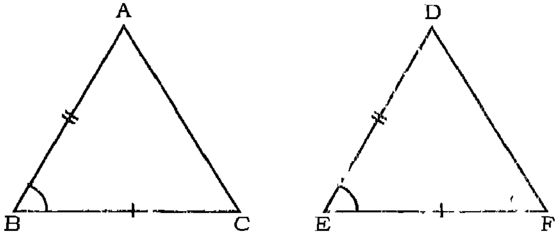


Fig. 11.11

for ΔABC to be congruent to ΔDEF, where AB = DE and BC = EF, we need to have ∠B (included between sides AB and BC) = ∠E (included between sides DE and EF).

Example 4 : In Fig. 11.12, AB = AD and ∠BAC = ∠DAC. Find the third pair of corresponding parts to make ΔABC ≅ ΔADC by SAS congruence condition.

Solution : In ΔABC and ΔADC,

AB = AD (Given)

and included angles ∠BAC = ∠DAC (Given)

Therefore, to make the two triangles congruent by SAS congruence condition, we must have AC = AC.

Thus, the required third pair of corresponding parts are AC and AC, i.e., the common side AC.

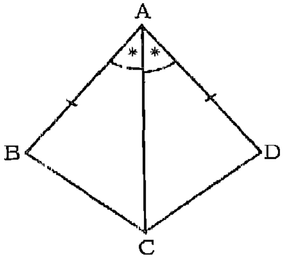


Fig. 11.12

EXERCISE 11.1

1. In ΔDEF and ΔYXZ, $DE = YX$ and $DF = YZ$ (Fig 11.13)

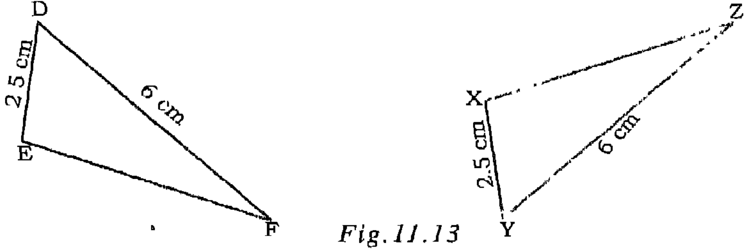


Fig. 11.13

What additional information is required to make the two triangles congruent by SAS congruence condition?

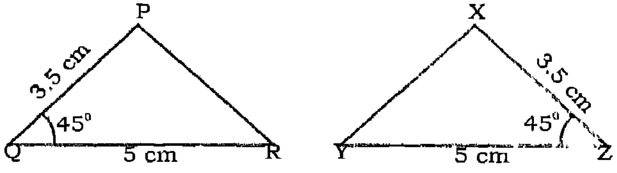
2. If for ΔABC and ΔDEF, the correspondence $CAB \leftrightarrow EDF$ gives a congruence, then which of the following statements are correct?

(i) $AC = DE$ (ii) $\angle A = \angle F$ (iii) $\angle B = \angle F$
(iv) $BC = DE$ (v) $AB = EF$ (vi) $\angle C = \angle E$
(vii) $AB = DF$ (viii) $\angle A = \angle D$

3. Let the correspondence $QPR \leftrightarrow ZYX$ give a congruence for ΔPQR and ΔXYZ. Fill in the blanks to make each statement true :

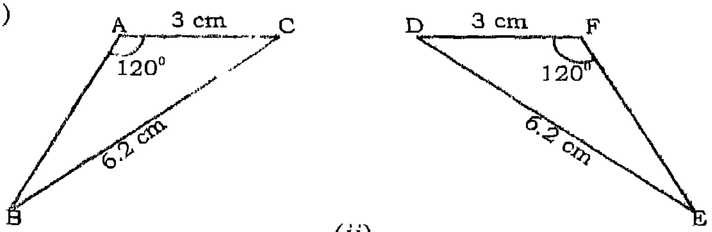
(i) $\angle R =$ ______ (ii) $QR =$ ________ (iii) $\angle P =$ _______
(iv) $QP =$ ______ (v) $\angle Q =$ ________ (vi) $RP =$ _______

4. In Fig 11.14, which pairs of triangles are congruent by SAS congruence condition? If congruent, write the congruence of the two triangles in symbolic form.

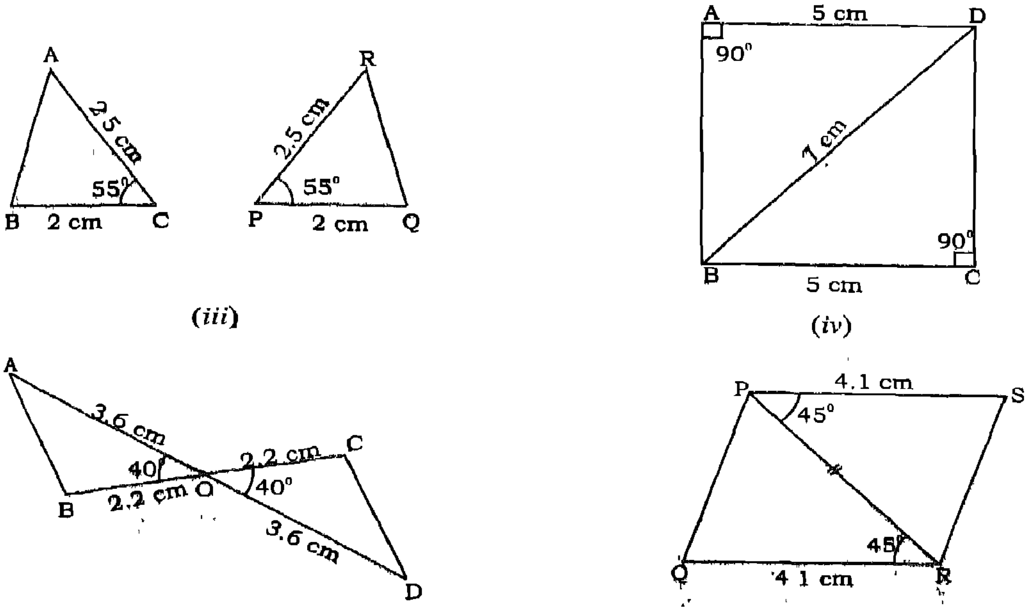


(*i*)

[*Hint:* $\Delta PQR \cong \Delta XZY$]



(*ii*)



(iii) (iv)

(v) (vi)

Fig. 11.14

5. ΔABC is isosceles with AB = AC (Fig. 11.15). Line segment AD bisects ∠A and meets the base BC in D. Find the third pair of corresponding parts which make

$$\Delta ADB \cong \Delta ADC$$

by SAS congruence condition.
Is it true to say that BD = DC?

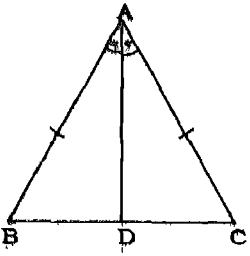


Fig. 11.15

6. In Fig. 11.16, AB ∥ DC and AB = DC.
   - (i) Is ∠BAC = ∠DCA? Why?
   - (ii) Is ΔABC ≅ ΔCDA by SAS congruence condition?
   - (iii) State the three facts you have used to answer (ii).

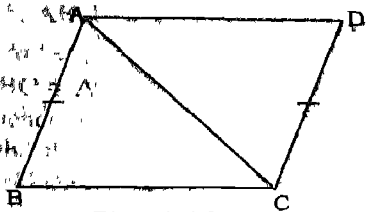


Fig. 11.16

7. In Fig.11.17, AB = BD and line segment BC bisects ∠ABD Which of the following statements is true by SAS congruence and why?

   (i) ΔABC ≅ ΔDCB.

   (ii) ΔABC ≅ ΔBCD.

   (iii) ΔABC ≅ ΔDBC

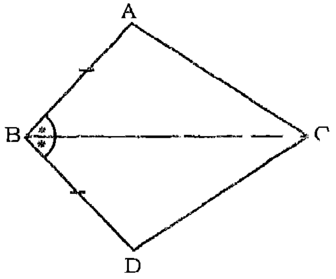


Fig 11.17

11.4.2 The 'Angle Side Angle' (ASA) Congruence Condition

Activity 3 : Draw a triangle ABC with BC = 6cm, ∠B = 50° and ∠C = 35°. Draw another ΔDEF with EF = 6cm, ∠E = 50° and ∠F = 35°. Then we have

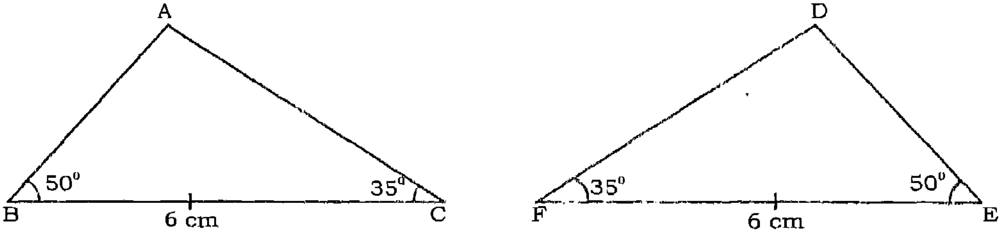


Fig 11.18

BC = EF, ∠B = ∠E and ∠C = ∠F (Fig.11.18). Make a trace copy of ΔDEF and place it on ΔABC. What do we observe?

We observe that with vertices D on A, E on B and F on C, ΔDEF exactly covers ΔABC. Similarly, if we place ΔABC on ΔDEF, we observe that ΔABC exactly covers ΔDEF, i.e., under the correspondence ABC ⟷ DEF the two triangles cover each other completely. So we have

$$\Delta ABC \cong \Delta DEF$$

Activity 4 : Construct a ΔABC with AB = 4cm, ∠A = 75° and ∠B = 45°. Construct another ΔPQR with PQ = 4cm, ∠P = 75° and ∠Q = 45°. Then, we have AB = PQ,

$\angle A = \angle P$ and $\angle B = \angle Q$ (Fig. 11.19).

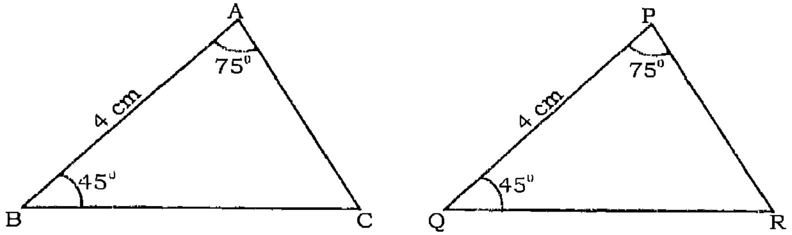


Fig. 11.19

Measure the remaining parts of the two triangles and record your observations in the form of a table

Remaining parts of ΔABC	*Corresponding parts of* ΔPQR	*Differences*
$\angle C =$	$\angle R =$	$\angle C - \angle R =$
AC =	PR =	AC − PR =
BC =	QR =	BC − QR =

You will observe that the differences ($\angle C - \angle R$), (AC − PR), and (BC − QR) are either zero or so small that the same may be treated as zero. Thus,

$$\angle C = \angle R,\ AC = PR \text{ and } BC = QR$$

Therefore, $\Delta ABC \cong \Delta PQR$.

Observe that in both the activities, we have taken two triangles such that two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle. In each case, we have found that the triangles were congruent. These two activities illustrate the following proposition .

Two triangles are congruent, if two angles and the included side of one triangle are respectively equal to the two angles and the included side of the other triangle

This is called ASA *congruence condition.*

Remark : If two angles of a triangle are given, then by angle sum property of a triangle, the third angle can be determined. Thus, whenever *one side and any two angles of a triangle are equal to the corresponding side and* *any two angles of another triangle,* then we can change this to '*two angles and the included side*' type, so that ASA congruence condition can be applied.

Example 5 : In Fig. 11.20, ΔPQR and ΔDQR are on the same base QR. State the equality between any three corresponding parts to ensure that ΔPQR ≅ ΔDRQ by ASA congruence condition.

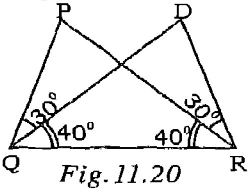


Fig. 11.20

Solution: In ΔPQR and ΔDRQ,

$$\angle PQR = \angle DRQ = 70^\circ \text{ (Given)}$$

$$\angle PRQ = \angle DQR = 40^\circ \text{ (Given)}$$

and QR is common to both triangles. Thus, the correspondence PQR ⟷ DRQ gives a congruence. Therefore, by ASA congruence condition,

$$\Delta PQR \cong \Delta DRQ.$$

Example 6: In Fig. 11.21, find the three pairs of corresponding parts to ensure that ΔABO ≅ ΔPQO by ASA congruence condition.

Solution: In ΔABO and ΔPQO,

$\angle AOB = \angle POQ = 40^\circ$ (Given) (1)

$\angle ABO = \angle PQO = 55^\circ$ (Given)

∴ by angle sum property, we have

$\angle BAO = \angle QPO = 85^\circ$ (2)

Also, AO = OP = 2.5 cm (Given) (3)

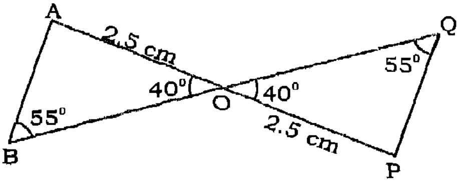


Fig. 11.21

From (1), (2) and (3), the correspondence ABO ⟷ PQO gives a congruence. Therefore, by ASA congruence condition,

$$\Delta ABO \cong \Delta PQO.$$

EXERCISE 11.2

1. In Fig. 11.22, $\angle A = \angle E = 55^\circ$ and $\angle B = \angle F = 60^\circ$. Find the third pair of corresponding parts which make ΔABC ≅ ΔEFD, by ASA congruence condition!

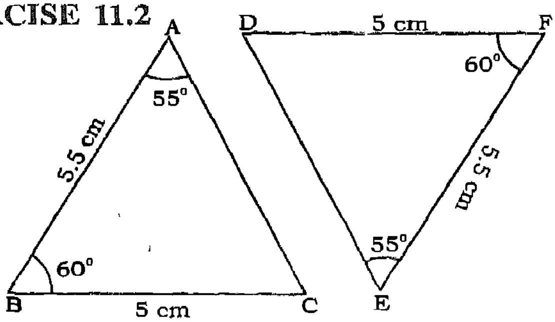
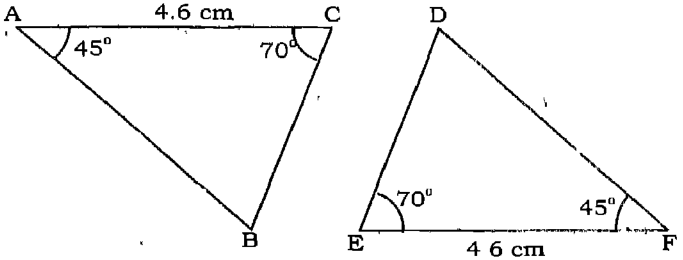
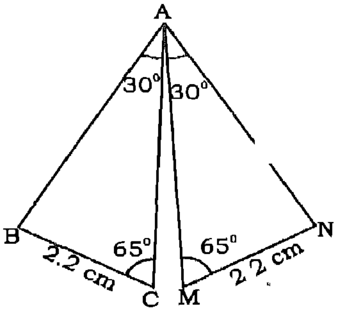


Fig. 11.22

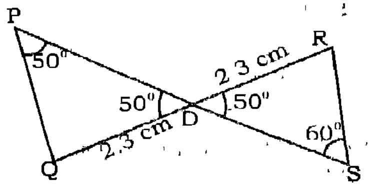
2. In Fig. 11.23, which pairs of triangles are congruent by ASA congruence condition? If congruent, write the congruence of the two triangles in symbolic form.



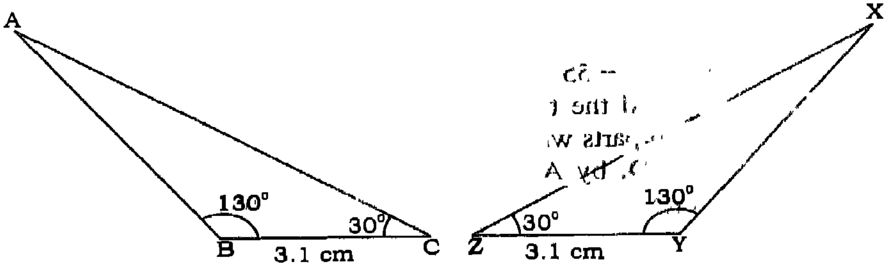
(i)



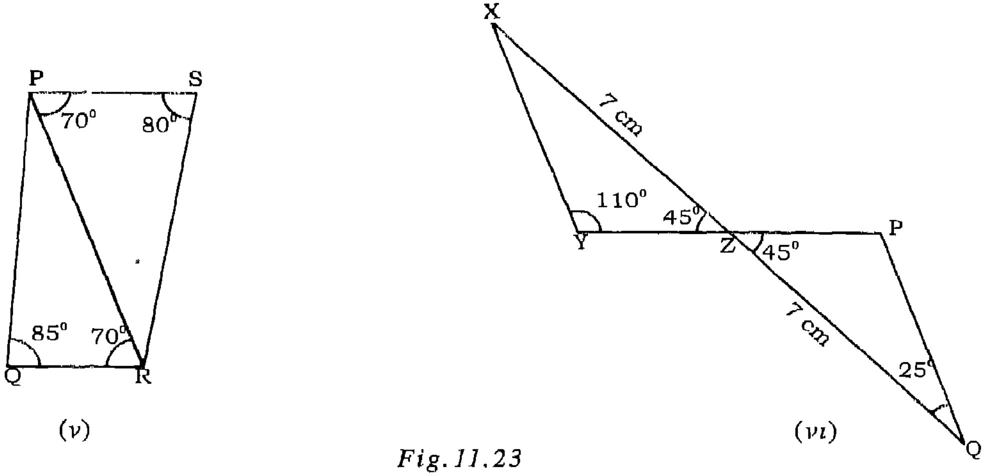
(ii)



(iii)



(iv)



(v)

(vi)

Fig. 11.23

3. In Fig. 11.24, AX bisects ∠BAC and ∠BDC. Find the third pair of corresponding parts to ensure that

$$\Delta ABD \cong \Delta ACD$$

by ASA congruence condition.

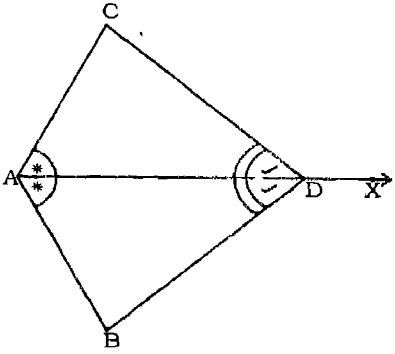


Fig. 11.24

4. In Fig. 11.25, AO = BO and ∠A = ∠B.
   - (i) Is ∠AOC = ∠BOD? Why?
   - (ii) Is ΔAOC ≅ ΔBOD by ASA congruence condition?
   - (iii) State the three facts you have used to answer (ii).
   - (iv) Is ∠ACO = ∠BDO? Why?

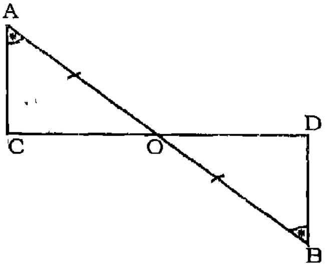


Fig. 11.25

5. In Fig 11.26, AD bisects ∠A and AD ⊥ BC.
   - (i) Is ΔADB ≅ ΔADC by ASA congruence condition?
   - (ii) If yes, state the three facts you have used to answer (i)
   - (iii) Is BD = DC? Why?

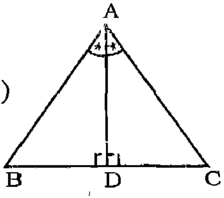


Fig. 11 26

11.4.3 The 'Side Side Side' (SSS) Congruence Condition

Activity 5 : Draw a ΔABC with AB = 4.5cm, BC = 6cm and CA = 3cm. Draw another ΔDEF with DE = 4.5cm, EF = 6cm and FD = 3cm. Then we have AB = DE, BC = EF and CA = FD (Fig. 11.27).

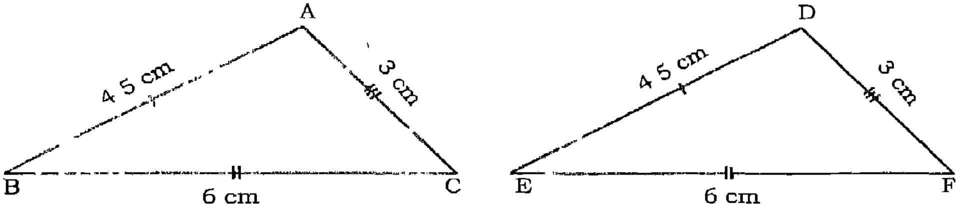


Fig. 11 27

Make a trace copy of ΔDEF and place it over ΔABC What do we observe? We observe that with vertex D on A, E on B and F on C, ΔDEF completely covers ΔABC, i.e., under the correspondence ABC ⟷ DEF, each triangle can be completely covered by the other. Therefore, we have

$$\Delta ABC \cong \Delta DEF.$$

Activity 6 : Draw a ΔABC with AB = 4cm, BC = 7cm and CA = 5cm. Draw another ΔPQR with PQ = 4cm, QR = 7cm and RP = 5cm. Then, we have AB = PQ, BC = QR and CA = RP (Fig.11 28).

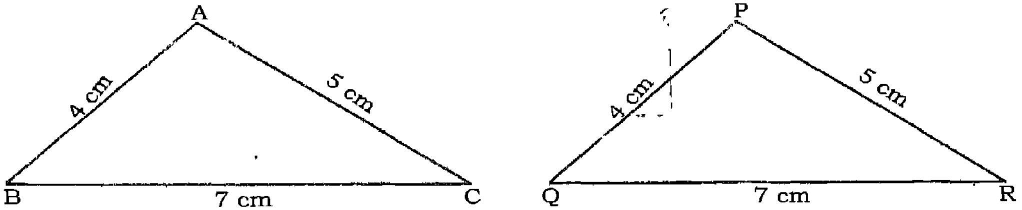


Fig. 11 28

Now, measure the corresponding angles of the two triangles and record your observations in the form of a table.

Angles of ΔABC	*Corresponding angles of* ΔPQR	*Differences*
$\angle A =$ $\angle B =$ $\angle C =$	$\angle P =$ $\angle Q =$ $\angle R =$	$\angle A - \angle P =$ $\angle B - \angle Q =$ $\angle C - \angle R =$

We observe that the differences ($\angle A - \angle P$), ($\angle B - \angle Q$) and ($\angle C - \angle R$) are either zero or so small that the same may be treated as zero. Therefore, we have

$$\angle A = \angle P,\ \angle B = \angle Q \text{ and } \angle C = \angle R$$

Since all the sides and angles of ΔABC are equal to the corresponding sides and angles of ΔPQR, we have

$$\Delta ABC \cong \Delta PQR.$$

Observe that in both the activities, we have taken two triangles such that the three sides of one triangle are equal to the three corresponding sides of another triangle. In each case, the two triangles are found to be congruent. These situations are special cases of the following proposition·

Two triangles are congruent, if the three sides of one triangle are respectively equal to the three corresponding sides of the other triangle. This is also called the SSS *congruence condition.*

Example 7 : In Fig.11.29, AB = DC and AD = BC. Find the third pair of corresponding parts which make $\Delta ABC \cong \Delta CDA$ by SSS congruence condition.

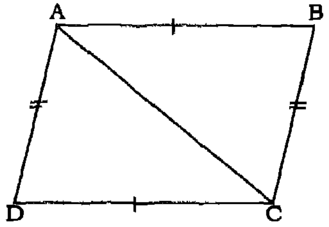


Fig. 11.29

Solution : In Fig. 11.29, we have two triangles, namely ΔABC and ΔADC. Two pairs of corresponding parts are

$$AB = DC \quad \text{(Given)}$$
$$BC = DA \quad \text{(Given)}$$

For SSS congruence condition, the third pair of corresponding parts are the remaining sides of the two triangles.

Clearly, AC is common to ΔABC and ΔCDA.

Thus, the third pair of correrponding parts are AC and CA.

Note: Under the correspondence $ABC \longleftrightarrow CDA$, $\Delta ABC \cong \Delta CDA$.

EXERCISE 11.3

1. In ΔABC and ΔDEF, $AB = DF$ and $BC = EF$ (Fig. 11.30).

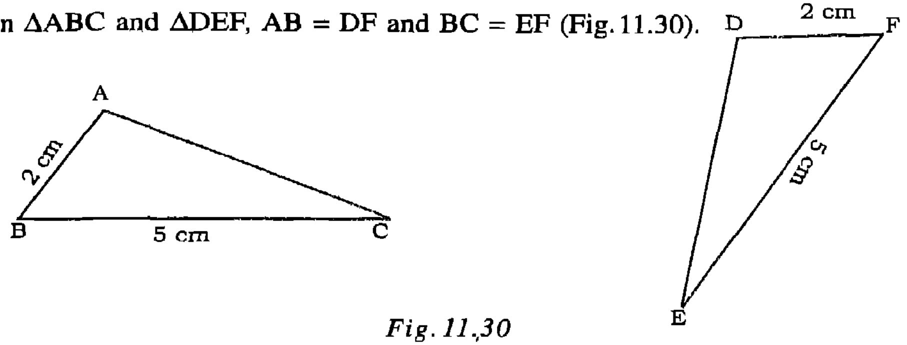
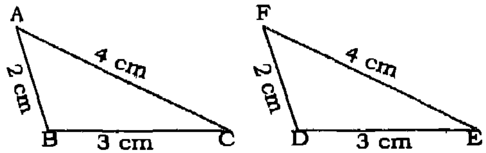


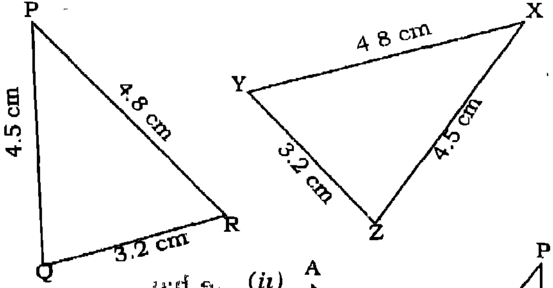
Fig. 11.30

What additional information is required to make the two triangles congruent by SSS congruence condition ?

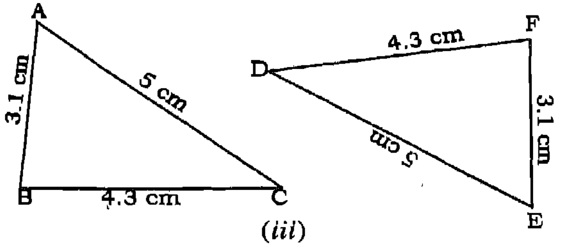
2. In Fig. 11.31, state which pairs of triangles are congruent by SSS congruence condition. If congruent, write the congruence of triangles in symbolic form



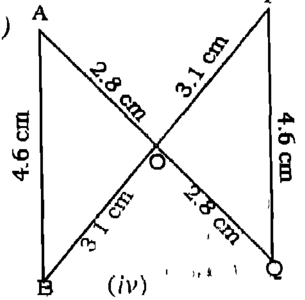
(*i*)



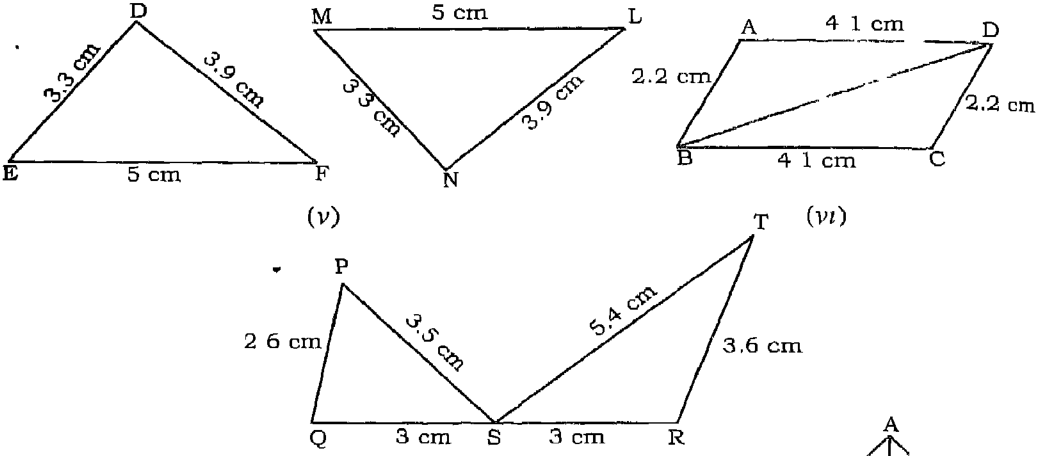
(*ii*)



(*iii*)



(*iv*)



(v) (vi)

(vii)

Fig 11 31

3. In Fig. 11.32, AB = AD and BC = DC. Find the third pair of corresponding parts that makes

$$\Delta ABC \cong \Delta ADC$$

by SSS congruence condition.

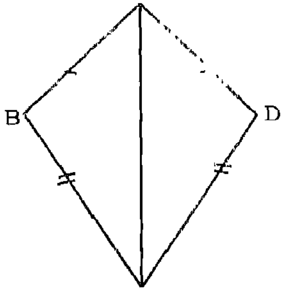


Fig 11 32

4. In Fig. 11.33, ΔABC is isosceles with AB = AC AD is the median from A to the opposite side BC. Find the third pair of corresponding parts to establish ΔADB ≅ ΔADC by SSS congruence condition

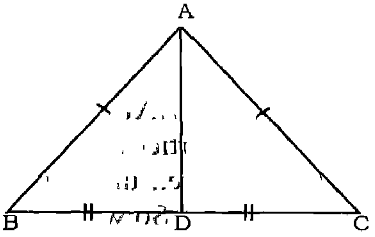


Fig. 11.33

5. ΔPQR and ΔTQR (Fig. 11.34) are on the same base QR. Also, PQ = TR and PR = TQ. Which of the following statements is true?

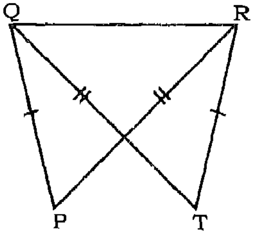


Fig. 11.34

(i) $\Delta PQR \cong \Delta TQR$
(ii) $\Delta PQR \cong \Delta TRQ$.
(iii) $\Delta PQR \cong \Delta RQT$.

11.4.4 The 'Right Angle Hypotenuse Side' (RHS) Congruence Condition

Activity 7 : Draw a ΔABC, right angled at B, with AB = 5cm and hypotenuse AC = 13cm. Draw another ΔPQR, right angled at Q, with PQ = 5cm and hypotenuse PR = 13cm. Then, we have $\angle B = \angle Q = 90°$, side AB = side PQ and hypotenuse AC = hypotenuse PR (Fig. 11 35)

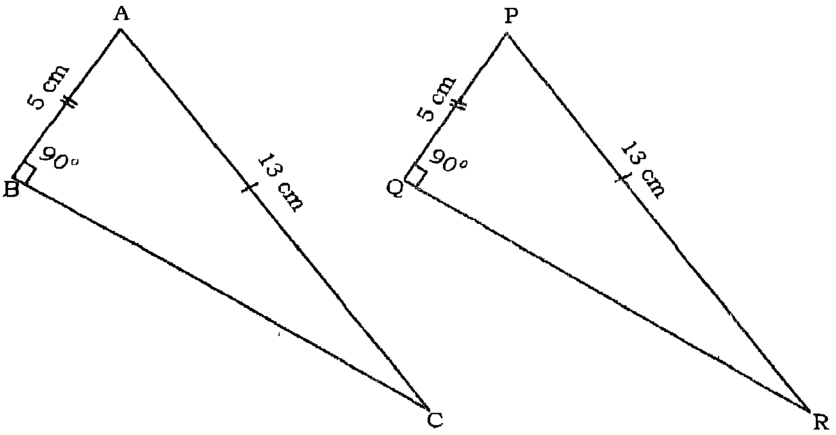


Fig. 11 35

Make a trace copy of ΔPQR and place it over ΔABC

What do we observe? We observe that with vertices P on A, Q on B and R on C, the two triangles cover each other exactly, i.e., under the correspondence ABC ⟷ PQR, ΔPQR completely covers ΔABC. So we have

$$\Delta ABC \cong \Delta PQR.$$

Activity 8 : Draw a ΔABC with $\angle A = 90°$, hypotenuse BC = 5cm and side AB = 3cm. Draw another ΔDEF with $\angle D = 90°$, hypotenuse EF = 5cm and side DE = 3cm. Then, we have

$\angle A = \angle D = 90°$, hypotenuse BC = hypotenuse EF and side AB = side DE (Fig. 11.36).

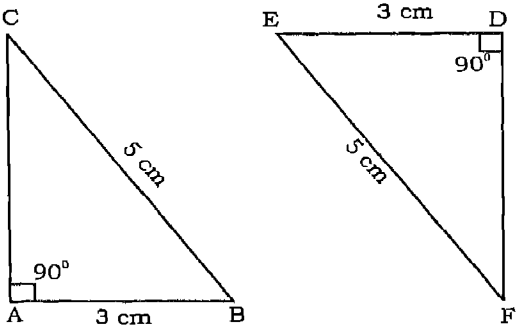


Fig 11.36

Measure the remaining parts of the two triangles and record your observations in the form of a table.

Remaining parts of ΔABC	*Corresponding parts of* ΔDEF	*Differences*
$\angle B =$	$\angle E =$	$\angle B - \angle E =$
$\angle C =$	$\angle F =$	$\angle C - \angle F =$
$AC =$	$DF =$	$AC - DF =$

We observe that the differences ($\angle B - \angle E$), ($\angle C - \angle F$) and (AC – DF) are either zero or so small that the same may be treated as zero. Thus,

$$\angle B = \angle E,\ \angle C = \angle F \text{ and side } AC = \text{side } DF.$$

Therefore, we may say that

$$\Delta ABC \cong \Delta DEF.$$

Observe that, in both the activities, we have taken two right triangles such that the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and the corresponding side of the other triangle. In each case, we have found that the two triangles are congruent. These situations are special cases of the following proposition :

Two right triangles are congruent, if the hypotenuse and one side of one triangle are respectively equal to the hypotenuse and the one side of the other triangle.

This is called RHS *congruence condition.*

Example 8 : In Fig. 11.37, AC = BD, DA ⊥ AB and CB ⊥ AB.

State any three facts needed to ensure that

$$\Delta ABC \cong \Delta BAD$$

by RHS congruence condition,

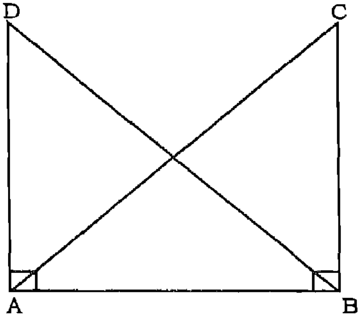


Fig 11 37

Solution : In ΔABC and ΔBAD,

hypotenuse AC = hypotenuse BD (Given)

$\angle ABC = \angle BAD = 90°$ (Given)

Also, side AB is common in both the triangles.

Thus, the correspondence ABC ⟷ BAD gives a congruence.

∴ ΔABC ≅ ΔBAD, by RHS congruence condition.

EXERCISE 11.4

1. In Fig. 11.38, $\angle B = \angle X = 90°$, and side BC = side XZ. What additional information is required to make

   $$\Delta ABC \cong \Delta YXZ$$

   by RHS congruence condition?

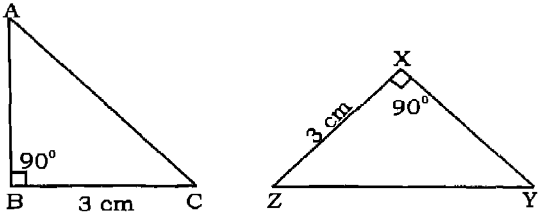
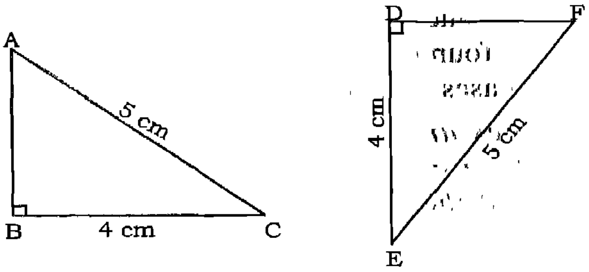
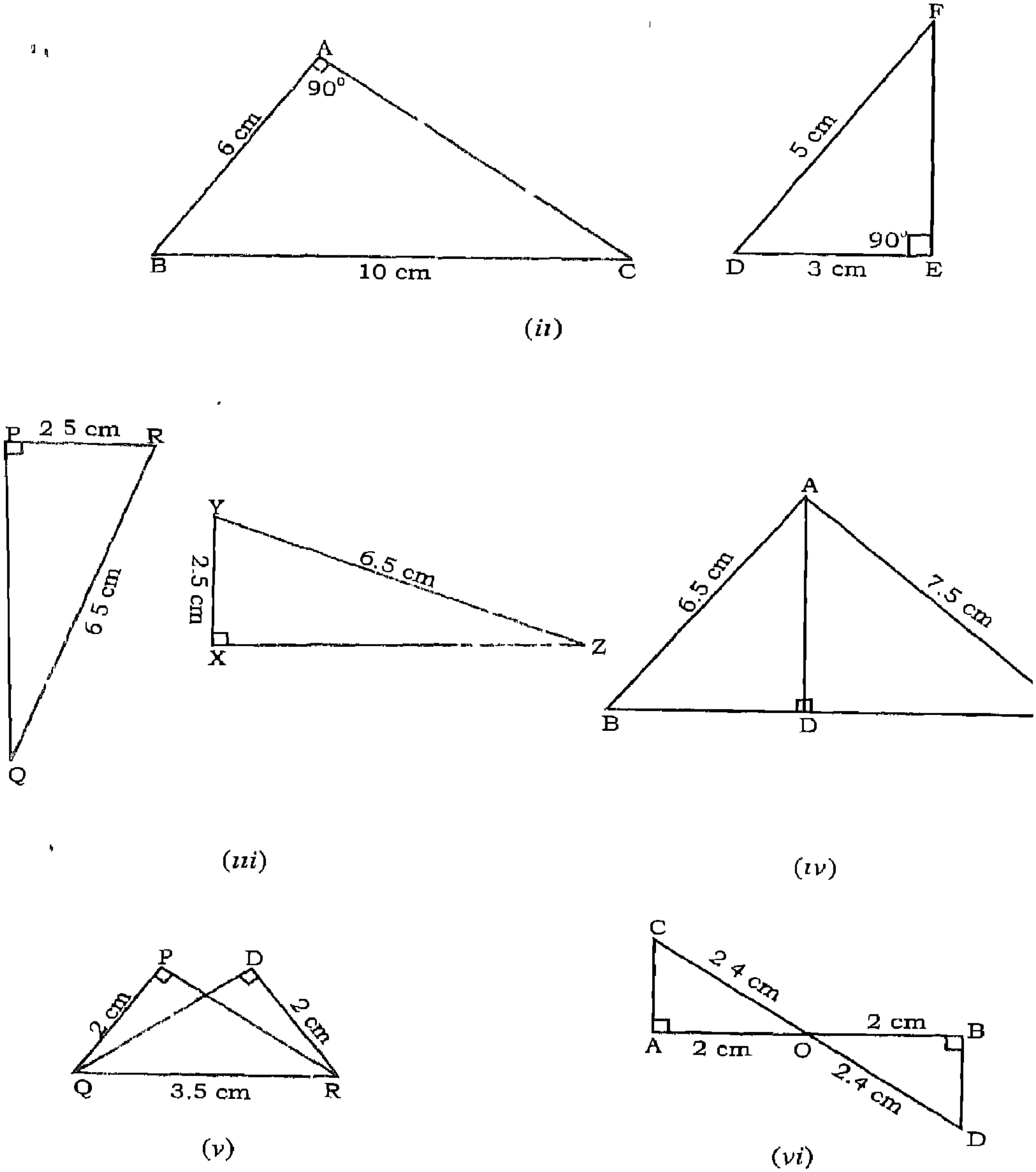


Fig. 11 38

2. In Fig. 11.39, state which pairs of triangles are congruent by RHS congruence condition. If congruent, write the congruence of triangles in symbolic form.



(*i*)



(ii)

(iii)

(iv)

(v)

(vi)

Fig. 11.39

3. In Fig. 11.40, AB = AD, AD⊥CD and AB⊥BC
   (i) Find the third pair of corresponding parts so that ΔABC ≡ ΔADC by RHS congruence condition
   (ii) Is BC = DC? Why?

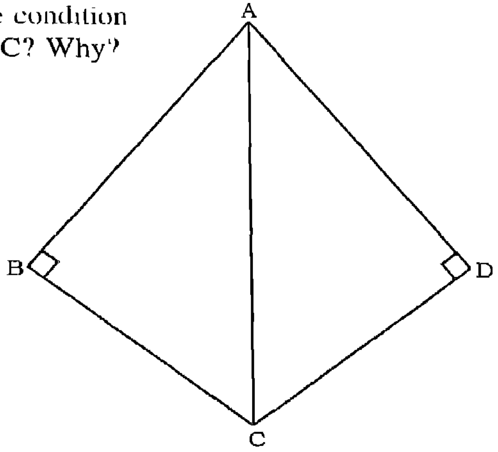


Fig. 11 40

4. ΔABC is isosceles with AB = AC AD is the altitude from A to side BC (Fig.11.41)
   (i) Is ΔABD ≡ ΔACD by RHS congruence condition?
   (ii) State the pairs of corresponding parts you have used to answer (i)
   (iii) Is BD = DC? Why?

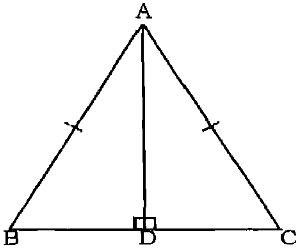


Fig. 11.41

5. In ΔABC, the altitudes BD and CE are equal (Fig. 11.42). Find the three pairs of corresponding parts which make ΔBCD ≡ ΔCBE by RHS congruence condition.

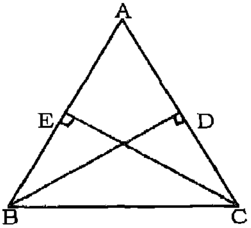


Fig. 11 42

MISCELLANEOUS EXERCISE

1. Fill in the blanks :

   (a) If $\Delta ABC \cong \Delta FDE$, then

   (i) $AB =$ ______ (ii) $BC =$ ______

   (iii) $AC =$ ______ (iv) $\angle A =$ ______

   (v) $\angle B =$ ______ (vi) $\angle C =$ ______

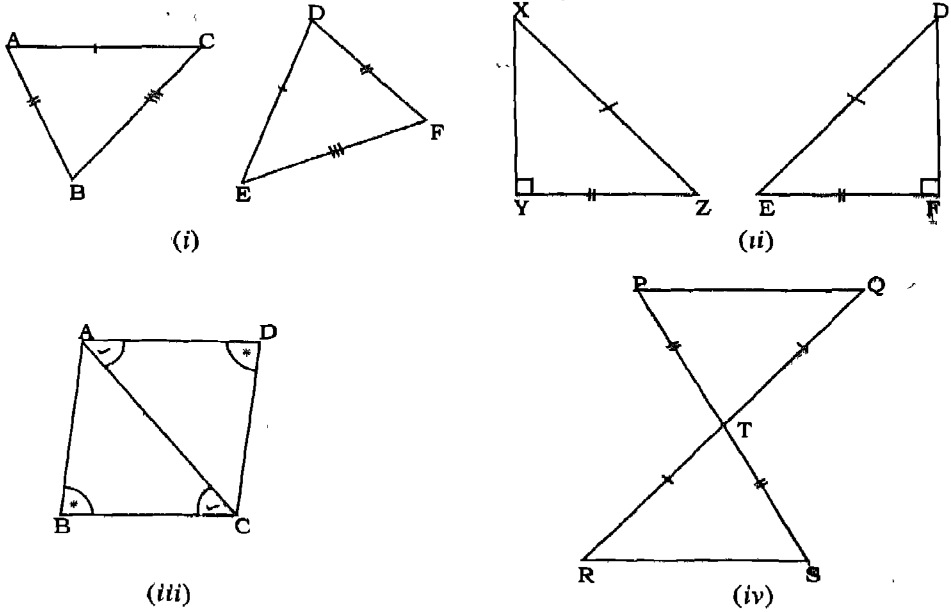
   (b) In ΔPQR, the angle included between sides PR and QR is ______ .

   (c) In ΔDEF, the side included between $\angle E$ and $\angle F$ is ______ .

   (d) If $AB = QP$, $\angle B = \angle P$, $BC = PR$, then by ______ congruence condition, $\Delta ABC \cong \Delta QPR$.

   (e) If $\angle A = \angle R$, $\angle B = \angle P$, $AB = RP$, then by ______ congruence condition, $\Delta ABC \cong \Delta RPQ$

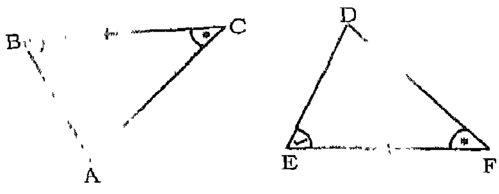
2. In Fig. 11.43, the pairs of corresponding equal parts in a pair of triangles are shown with similar markings. Specify, by which condition, the two triangles become congruent Also, determine the correspondence between the pairs of triangles which gives the congruence.



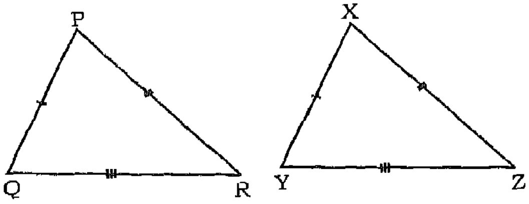
(i) (ii) (iii) (iv)

Fig. 11.43

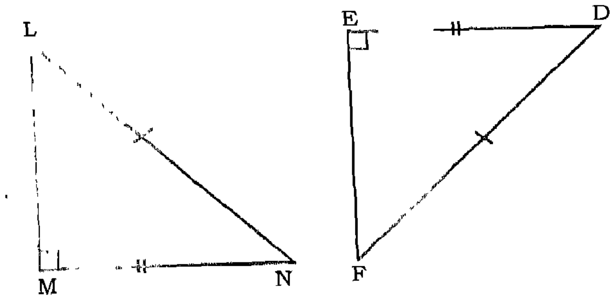
3. In Fig 11.44, each pair of triangles has markings indicating equal parts. Name the congruence condition (SAS, ASA, SSS, RHS), if any, that will establish the two triangles of a pair to be congruent



(i)



(ii)



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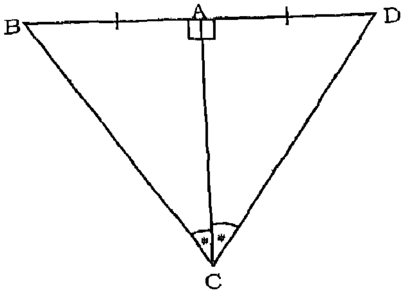
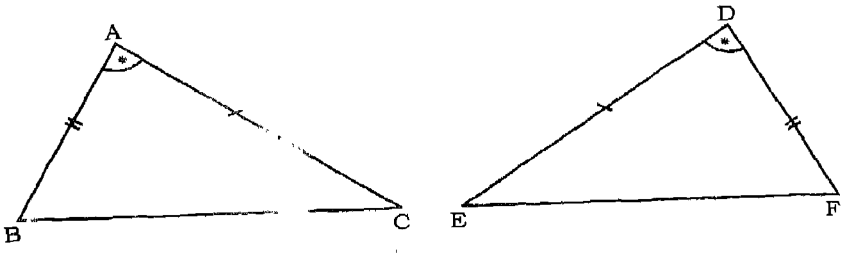
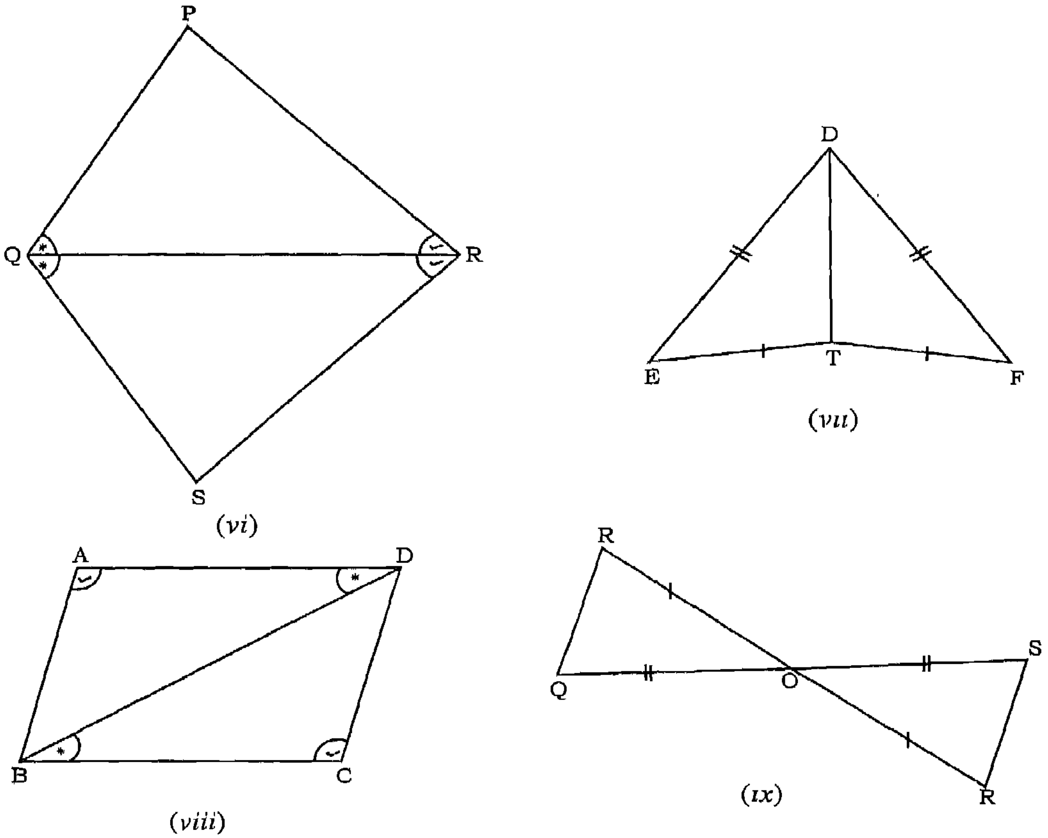


(iv)



(v)



(*vi*) (*vii*) (*viii*) (*ix*)

Fig. 11 44

4. In Fig. 11.45, ABC and DBC are two triangles on a common base BC such that AB = DC and DB = AC, where A and D lie on the same side of BC. In ΔADB and ΔDAC, state the corresponding parts so that

   ΔADB ≅ ΔDAC

   Which condition do you use to establish the congruence? If ∠DCA = 40° and ∠BAD = 100°, then find ∠ADB.

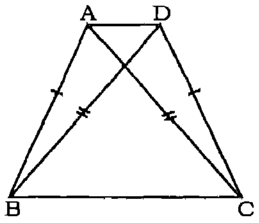


Fig 11.45

Things to Remember

1. Two figures are congruent, if they have the same shape and size.
2. Two triangles are congruent, if in a matching of their vertices, the three sides and the three angles of one triangle are respectively equal to the corresponding parts of the other
3. Two triangles are congruent, if two sides and the included angle of the one triangle are respectively equal to the two sides and the included angle of the other (SAS congruence condition).
4. Two triangles are congruent, if two angles and the included side of the one triangle are respectively equal to the two angles and the included side of the other (ASA congruence condition).
5. Two triangles are congruent, if the three sides of one triangle are respectively equal to the three sides of the other (SSS congruence condition)
6. Two right triangles are congruent, if the hypotenuse and one side of the one triangle are respectively equal to the hypotenuse and one side of the other (RHS congruence condition)

QUADRILATERALS

CHAPTER 12

12.1 Introduction

Quadrilaterals are geometric figures that are commonly found in our surroundings. If you look around, you will find a quadrilateral in one form or the other such as, square, rectangle, parallelogram, trapezium, etc. To recognize and understand the properties of these figures, let us learn some basic concepts about quadrilaterals.

12.2 Quadrilaterals

You may look around while sitting in the classroom and observe that the table-top, the ceiling, the door, the floor, etc. all suggest quadrilaterals. Following these observations, we describe a quadrilateral in mathematical terms as a four-sided simple closed figure lying in a plane. Some examples are :

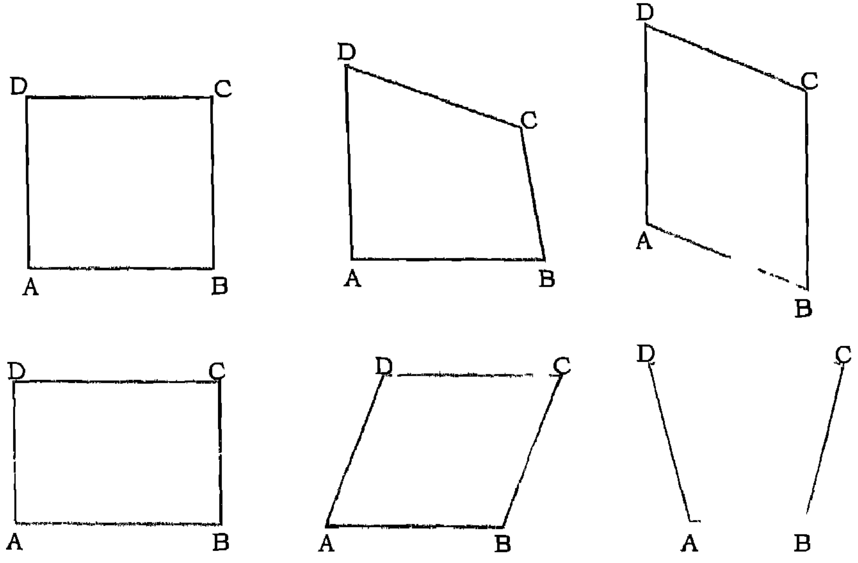


Fig. 12.1

But the figure given below is not a quadrilateral. Even though it is a four-sided closed figure, it is not simple.

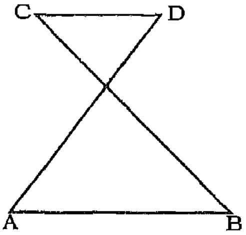
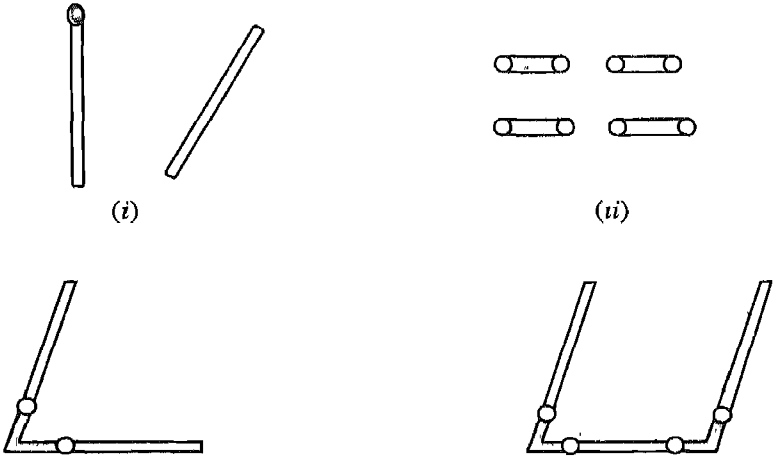


Fig 12.2

Activity :

1. Take a number of match sticks and chop off their phosphorus-coated ends [Fig. 12.3.(i)]. Take a number of small pieces of rubber tubes [Fig. 12.3 (ii)].



(*i*) (*ii*)

(*iii*) (*iv*)

Fig. 12 3

2. Join two match sticks with rubber tubing so that an angle is formed [Fig. 12.3 (iii)].
3. Join another match stick at one of the free ends of the two match sticks of step 2 so that no two match sticks are in the same line [Fig. 12.3 (iv)].

4. Join fourth match stick to both the free ends in such a way that no two match sticks are in the same line and no match stick intersect another except at the end points (Fig.12.4) By doing so, we obtain an object in the shape of a quadrilateral. But, unlike a triangle, shape of the quadrilateral is not rigid. It can easily be changed You can observe this yourself by pressing the opposite corners of the quadrilateral. To make it rigid, we have to fix two more match sticks on opposite corners

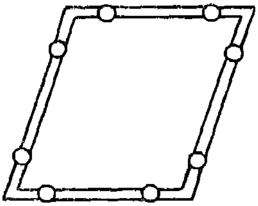


Fig.12.4

In mathematical terms, we explain a quadrilateral as follows

If we take four points A, B, C *and* D *in a plane so that no three of them are collinear and the line segments* AB, BC, CD *and* DA *intersect at their ends only, then the figure so formed by the four line segments* AB, BC, CD *and* DA *is called a quadrilateral* (Fig.12.5).

The points A, B, C and D are called *vertices* of the quadrilateral. We shall, for brevity, call the quadrilateral with vertices A, B, C and D as quadrilateral ABCD (Fig.12.5).

The line segments AB, BC, CD and DA are called the *sides* of the quadrilateral ABCD.

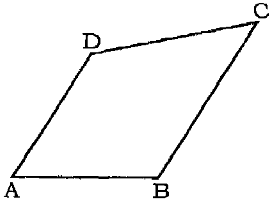


Fig. 12 5

The four angles formed by the line segments at the vertices viz. ∠DAB, ∠ABC, ∠BCD and ∠CDA are called the *angles* of the quadrilateral ABCD.

Notation: We shall often denote the four angles of quadrilateral ABCD by ∠A, ∠B, ∠C and ∠D which respectively stand for ∠DAB, ∠ABC, ∠BCD and ∠CDA.

12.2.1 Diagonals of a Quadrilateral

You learnt in the previous section that the shape of a quadrilateral can easily be changed, but if we join the vertices A and C, and B and D respectively by match sticks/line segments AC and BD, then the shape of the quadrilateral becomes rigid.

The two line segments AC *and* BD *joining opposite vertices are called diagonals of the quadrilateral* ABCD (Fig. 12.6).

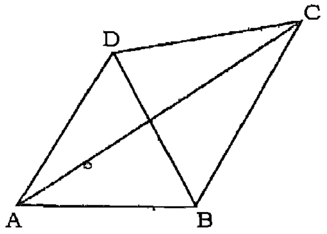


Fig 12.6

12.2.2 Adjacent Sides and Opposite Sides

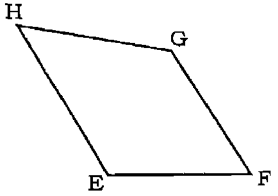


Fig 12 7

If we take any two sides of a quadrilateral, then either the two sides have a common vertex or they do not have a vertex in common. For example, the sides EF and EH have a vertex E in common (Fig. 12 7), but the sides EF and GH do not have any common vertex. Two sides that have a common vertex are known as *adjacent sides*. Any two sides that do not have a vertex in common are known as *opposite sides*. EF, GH and GF, HE are pairs of opposite sides of the quadrilateral EFGH. Pairs EF and FG; FG and GH; GH and HE ; HE and EF are pairs of adjacent sides.

12.2.3 Adjacent Angles and Opposite Angles

If we take any two angles of a quadrilateral, then either the two angles have a side in common or they do not have a side in common. For example, $\angle$A and $\angle$B have a side AB in common (Fig.12 8). Such angles are called *adjacent angles*.

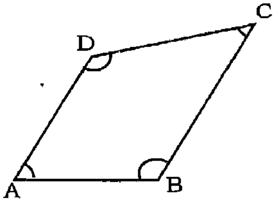


Fig.12.8

Similarly, the other pairs of adjacent angles are $\angle$B and $\angle$C , $\angle$C and $\angle$D and $\angle$D and $\angle$A. Thus, two angles of a quadrilateral are said to be *adjacent angles*, if they have a side of the quadrilateral as a common arm.

$\angle$B, $\angle$C are adjacent because the side BC is a common arm of $\angle$B and $\angle$C.

Similarly, the adjacent angles $\angle$C and $\angle$D have the side CD as a common arm. Also, the adjacent angles $\angle$D and $\angle$A have side DA as a common arm.

Two angles of the quadrilateral that are not adjacent are called *opposite angles*. Thus, ∠A, ∠C form a pair of opposite angles of quadrilateral ABCD (Fig. 12.8). ∠B and ∠D are also opposite angles.

12.2.4 The Interior and the Exterior of a Quadrilateral

We know that a quadrilateral is a four-sided figure lying in a plane. A quadrilateral divides the points of its plane into the following three parts :

(i) The part of the plane which contains all points such as P, Q, etc. that are enclosed by the quadrilateral [Fig.12.9(i)].

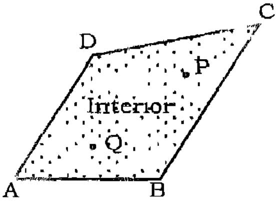


Fig. 12.9 (i)

This part is called the *interior* of the quadrilateral.

Points P and Q are in the interior of the quadrilateral ABCD.

We also say that P, Q are *interior points* of the quadrilateral ABCD

(ii) The part of the plane which contains all points such as R and T of the plane that are not enclosed by the quadrilateral [Fig.12.9(ii)].

This part is called the *exterior* of the quadrilateral. Points R and T are in the exterior of the quadrilateral ABCD. We also say that R and T are *exterior points* of quadrilateral ABCD

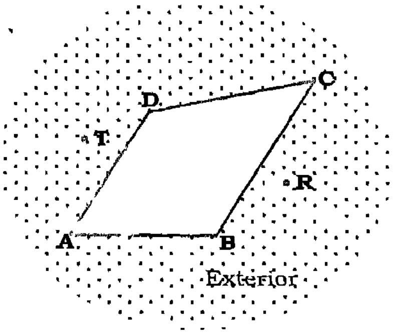
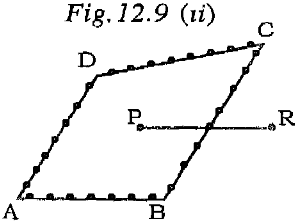


Fig. 12.9 (ii)

(iii) The quadrilateral ABCD itself [Fig.12.9 (iii)].

The points P and R lie respectively in the interior and the exterior of quadrilateral ABCD. To draw a line segment PR, we have to take some point common with quadrilateral ABCD.

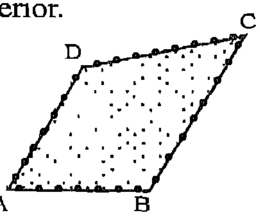


Quadrilateral

Fig 12 9 (iii)

Thus, it is not possible to go from the interior of a quadrilateral to its exterior or vice-versa, without crossing the quadrilateral ABCD at some point of it. Hence, quadrilateral ABCD is the *boundary* of its interior.

The interior of the quadrilateral ABCD, *together with its boundary, i.e., the quadrilateral itself, is called the quadrilateral region* ABCD [Fig.12.9(iv)]



Quadrilateral Region

Fig 12 9 (iv)

EXERCISE 12.1

1. Fill in the blanks
   - (i) A quadrilateral has________________vertices.
   - (ii) A quadrilateral has________________sides.
   - (iii) A quadrilateral has________________angles.
   - (iv) A quadrilateral has________________diagonals.
2. Draw a quadrilateral ABCD and the diagonal AC Into what figures does the diagonal divide it? How many such figures are formed?
3. Take a point O inside a given quadrilateral ABCD Join the point O to the vertices A, B, C and D. Into what figures will the quadrilateral be divided ? Name the figures.
4. In Fig. 12 10, PQRS is a quadrilateral
   - (i) How many pairs of adjacent sides are there? Name them.
   - (ii) How many pairs of opposite sides are there? Name them.
   - (iii) How many pairs of adjacent angles are there? Name them
   - (iv) How many pairs of opposite angles are there? Name them.

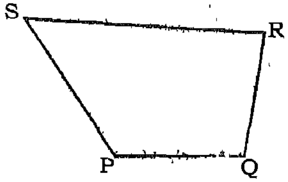


Fig.12.10

5. Draw a quadrilateral and label it ABCD. Shade the quadrilateral region. Name the (i) vertices, (ii) angles, (iii) diagonals, (iv) adjacent sides, (v) adjacent angles, (vi) opposite sides, and (vii) opposite angles of the quadrilateral.
6. Mark a point M in the interior and a point N in the exterior of quadrilateral ABCD. Join MN. Does MN meet the quadrilateral? If yes, in how many points?

12.3 Angle Sum Property of a Quadrilateral

Activity 1 :

1. Take a thick sheet of paper and draw a quadrilateral ABCD.

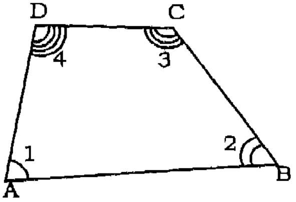


Fig.12 11

2. Mark the four angles and label them as 1, 2, 3 and 4. Cut-out the quadrilateral region ABCD along its sides with a pair of scissors (Fig.12.11).

3. Cut the cut-out region into four parts by lines such that each part determines one of the labelled angles of the quadrilateral (Fig.12.12).

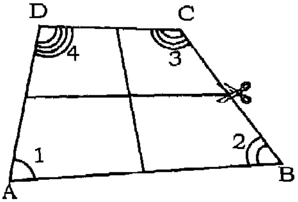


Fig.12.12

4. Draw a ray OP. Arrange the four cut-out angles, without an overlap, in such a way that the vertex of each angle falls on O. Start with ∠1, putting one of its arms along the ray OP (Fig. 12.13). Place the other angles one-by-one so that no two of them overlap and no gaps are left.

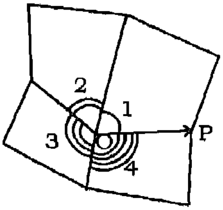


Fig.12.13

5. What do we find? We find that the four angles at O, together make up the complete angle. We know that measure of a complete angle is 360° This shows that the sum of the angles of the quadrilateral ABCD is 360°.

Activity 2 :

Draw four quadrilaterals. Label each of these as quadrilateral ABCD. Number these quadrilaterals as I, II, III and IV. Measure the angles of each of the quadrilaterals and record the observations in the table given below:

Quadrilateral	$\angle A, \angle B, \angle C, \angle D$	$S=\angle A+\angle B+\angle C+\angle D$	$360^\circ - S$
I			
II			
III			
IV			

The difference $360^\circ - S$ is either zero or so small that the same may be ignored. If the measurements are not correct, a non-zero value is obtained.

Thus, in all cases

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$

Activity 3 :

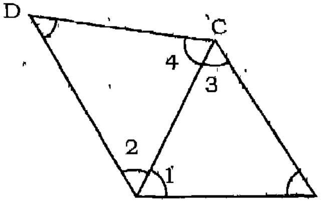


Fig. 12 14

1. Draw a quadrilateral ABCD. Draw the diagonal AC. The diagonal divides the quadrilateral ABCD into two triangles ABC and ADC (Fig. 12 14).
2. Label the angles BAC, DAC, BCA and DCA as 1, 2, 3 and 4 respectively
3. Can you find $\angle 1 + \angle B + \angle 3$ and $\angle 2 + \angle 4 + \angle D$? Yes, by using angle sum property of a triangle,

$$\angle 1 + \angle B + \angle 3 = 180^\circ \qquad (1)$$
$$\angle 2 + \angle 4 + \angle D = 180^\circ \qquad (2)$$

Adding (1) and (2), we have

$$\angle 1 + \angle B + \angle 3 + \angle 2 + \angle 4 + \angle D = 180^0 + 180^0 = 360^0$$

or $(\angle 1 + \angle 2) + \angle B + (\angle 3 + \angle 4) + \angle D = 360^0$

or $\angle DAB + \angle B + \angle BCD + \angle D = 360^0$

or $\angle A + \angle B + \angle C + \angle D = 360^\circ = 4$ right angles

i.e., *sum of the angles of a quadrilateral is 360^0 or 4 right angles*

Example 1 : $\angle D$ of the quadrilateral ABCD is equal to 120° and $\angle A = \angle B = \angle C$. Calculate the three angles of the quadrilateral.

Solution : In quadrilateral ABCD,

$$\angle D = 120^\circ.$$

Let $\angle A = \angle B = \angle C = x^\circ$.

Then by the angle sum property of a quadrilateral,

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$

or $x^\circ + x^\circ + x^\circ + 120^\circ = 360^\circ$

or $3x^\circ = 360^\circ - 120^\circ = 240^\circ$

or $x^\circ = 80^\circ$

Thus, $\angle A = \angle B = \angle C = 80^\circ$

Example 2 : Three angles of a quadrilateral are in the ratio 1 : 2 : 3. The sum of the least and the greatest of these angles is equal to 180°. Find all the angles of the quadrilateral.

Solution : Suppose that the measures of the three angles are x°, $2x^\circ$ and $3x^\circ$.

(As angles are in the ratio 1 : 2 : 3)

Then, $x^\circ + 3x^\circ = 180^\circ$ (By the given condition)

or $x^\circ = 45^\circ$

∴ Other angles are

$$2x^\circ = 2 \times 45^\circ = 90^\circ, \quad 3x^\circ = 3 \times 45^\circ = 135^\circ$$

Thus, three angles are 45°, 90° and 135°.

By the angle sum property of a quadrilateral,

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$

∴ $45^\circ + 90^\circ + 135^\circ + \angle D = 360^\circ$

or $\angle D = 360^\circ - 270^\circ = 90^\circ$

Hence, the angles of the quadrilateral ABCD are 45°, 90°, 135° and 90°.

EXERCISE 12.2

1. A quadrilateral has three angles measuring 70°, 50° and 125°. What is the measure of the fourth angle ?
2. In a quadrilateral ABCD, $\angle D$ is equal to 150° and $\angle A = \angle B = \angle C$. Find $\angle A$, $\angle B$ and $\angle C$.
3. Two angles of a quadrilateral are of measure 65° each. The third angle measures 135°. What is the measure of the fourth angle?
4. A quadrilateral has all four angles of the same measure. What is the measure of each?
5. Is it possible to have a quadrilateral whose angles are of measures 125°, 135°, 60° and 75° ? Give reasons.
6. The four angles of a quadrilateral are in the ratio 3 : 5 : 7 : 9. Find the angles separately.
7. The angles of a quadrilateral are in the ratio 1 : 2 : 3 . 4. What are the measures of the four angles?

Things to Remember

1. If we take four points say A, B, C and D in a plane such that no three of them are collinear and the line segments AB, BC, CD and DA intersect at their ends only, then the figure so formed by the four line segments is called a quadrilateral.
2. The points A, B, C and D are called the vertices of the quadrilateral ABCD
3. The line segments AB, BC, CD and DA are called the sides of the quadrilateral ABCD.
4. The four angles of the quadrilateral ABCD are $\angle A$, $\angle B$, $\angle C$ and $\angle D$.
5. The two line segments joining the opposite vertices are called the diagonals of the quadrilateral ABCD.
6. In a quadrilateral, two sides that have a common vertex are known as adjacent sides.
7. In a quadrilateral, any two sides that do not have a vertex in common are called opposite sides.
8. Two angles of a quadrilateral are adjacent angles, if they have a side of the quadrilateral as a common arm.
9. Two angles of the quadrilateral that are not adjacent are called opposite angles.
10. The interior of the quadrilateral ABCD, together with its boundary (i.e , the quadrilateral itself) is called the quadrilateral region ABCD.
11. The sum of the angles of a quadrilateral is 360°.

CIRCLES

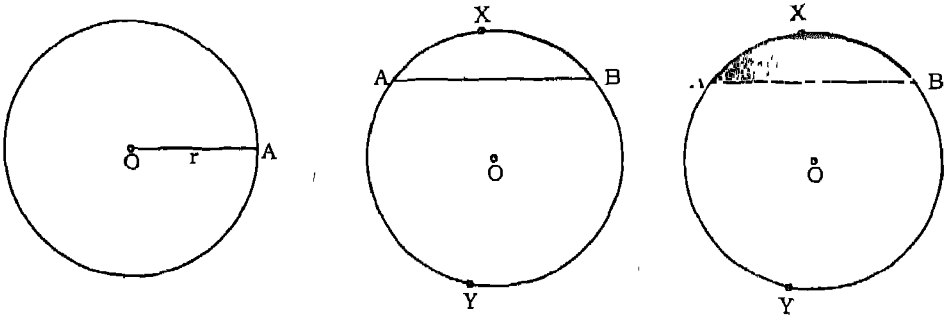
CHAPTER 13

13.1 Introduction

You have already read about circles in Class VI. Here, we shall review the concepts of circle, its centre and radius, semicircle, etc., and introduce the concept of a *segment of a circle*. We shall also study some properties of circles.

13.2 Segment of a Circle

Recall that a *circle* is a closed plane figure consisting of all those points of the plane which are at a constant distance from a fixed point O [Fig.13.1(i)].



(i) (ii) (iii)

Fig 13.1

The fixed point O is called the *centre* of the circle and the constant distance OA = r say, is called the *radius* of the circle

Let A and B be any two points on the circle [Fig.13.1 (ii)]. Join A and B. Line segment AB is called a *chord* of the circle. A chord passing through the centre of the circle is called a *diameter* of the circle. The points A and B divide the circle in two parts called the *arcs* of the circle. Usually the two arcs are unequal and are called *major and minor arcs*, e.g., major arc AYB and minor arc AXB in Fig.13.1 (ii).

The region bounded by an arc of the circle and the chord of the arc is called *a segment of the circle*. In Fig.13.1 (iii), the shaded portion AXB of the circle is segment AXB of the circle The unshaded portion AYB is another segment of the circle.

Note that AXB is the *minor segment* and AYB is the *major segment*.

13.3 Semicircular Region

We know that in the special case, when the chord is a diameter of the circle, its end-points divide the circle into two equal parts (arcs). Each part is called a *semicircle*. In Fig. 13.2 (i), AB is a diameter of the circle with centre O. Each part ACB and ADB is a semicircle [Fig 13. 2 (ii)]. Observe that A and B are the end-points of either semicircle. The diameter AB, except for the end-points, is not a part of either semicircle.

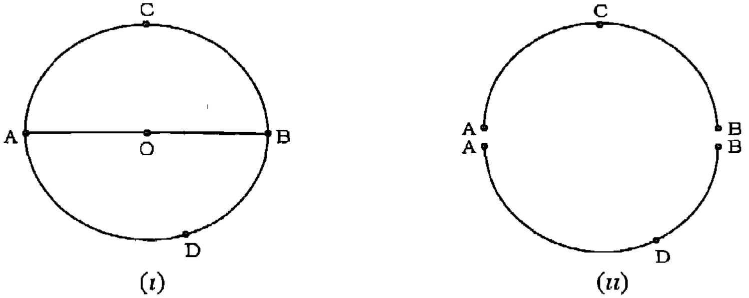


Fig 13.2

Let us consider the part of the plane that consists of diameter AB, semicircle ACB and interior of the circle that is enclosed by the semicircle ACB and the diameter AB. Shade this region as shown in [Fig.13.3 (i)]. We call it *semicircular region* ACB. However, the unshaded region ADB together with AB is also a semicircular region. The two semicircular regions ACB and ADB together make up a circular region ADBCA [Fig.13.3 (ii) and (iii)].

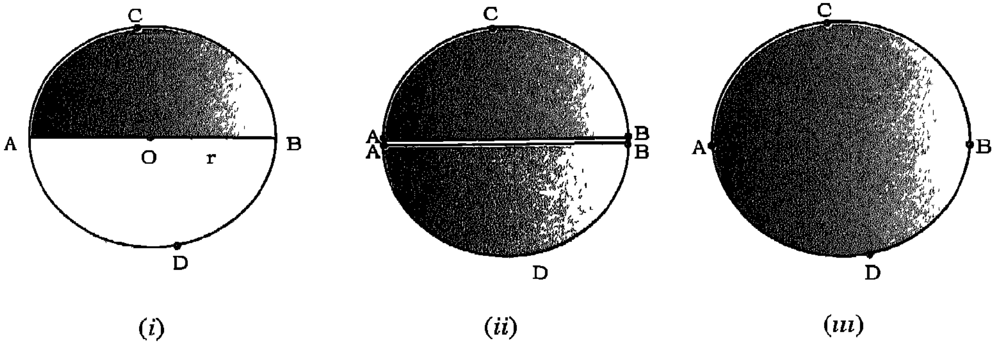


Fig. 13.3

13.4 Angles in a Semicircle and a Segment

Draw a circle with centre O. Draw a diameter PQ of the circle [Fig.13.4(i)]. Mark a point S on one of the semicircles. Now join PS and QS. ∠PSQ so formed is *an angle in a semi-circle.*

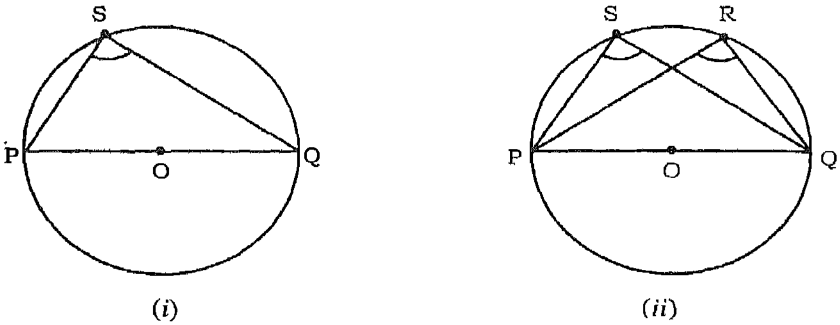


Fig 13.4

Thus, *an angle in a semicircle is an angle formed by two line segments drawn from any point of the semicircle to the end-points of its diameter.* In Fig.13.4 (i), ∠PSQ is an angle in a semicircle. Can you draw more angles in the same semicircle? Yes. In Fig.13.4 (ii), ∠PRQ is another angle in the same semicircle

13.4.1 Angles in a Segment

Consider a chord RT of a circle with centre O [Fig 13.5 (i)]. Recall that a segment of a circle is the region bounded by the chord and one of the two parts (arcs) into which the end-points of the chord divide a circle.

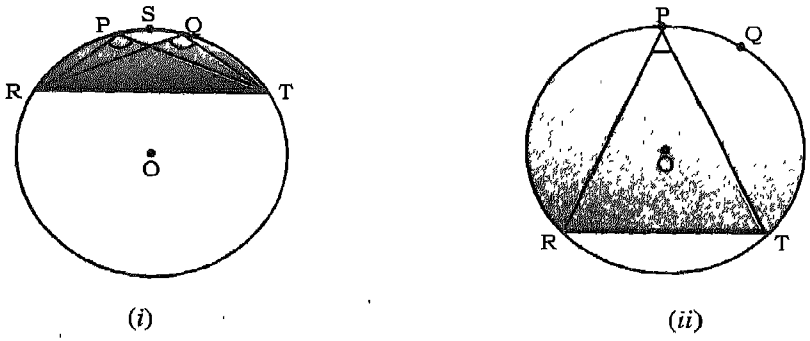


Fig. 13.5

Take the segment RST of the circle with centre O [Fig. 13.5(i)] and a point P on RST distinct from R and T. Join RP and PT to form the ∠RPT. We say that ∠RPT is *an angle in the segment* RST [Fig. 13.5 (i)] ∠RQT is also an angle in the same segment of the circle. Can you draw more angles in the same segment? If yes, how many more?

Similarly, ∠RPT is an angle in the segment formed by the major arc RQT [Fig. 13.5 (ii)].

Thus, *an angle in a segment is an angle, formed by two line segments drawn from any point in the arc of the segment to the end-points of its chord*

13.5 Properties of Angles Relating to Circles

Let us study some interesting properties of angles in a semicircle and angles in the same segment.

Activity :

1. Draw a line segment AB of any length. Mark the mid-point of AB as O.
2. Draw a semicircle on AB as diameter.
3. Take any point C on the semicircle. Join AC and BC. Thus, ∠ACB is an angle in the semicircle APB.
4. Now take a protractor and superpose its 0-180 line along AC so that the centre of the protractor falls on C (Fig.13.6). Note that the measure of the ∠ACB by looking at the marking on the protractor corresponding to arm CB of ∠ACB. What do you observe? ∠ACB is 90^0 or a right angle.

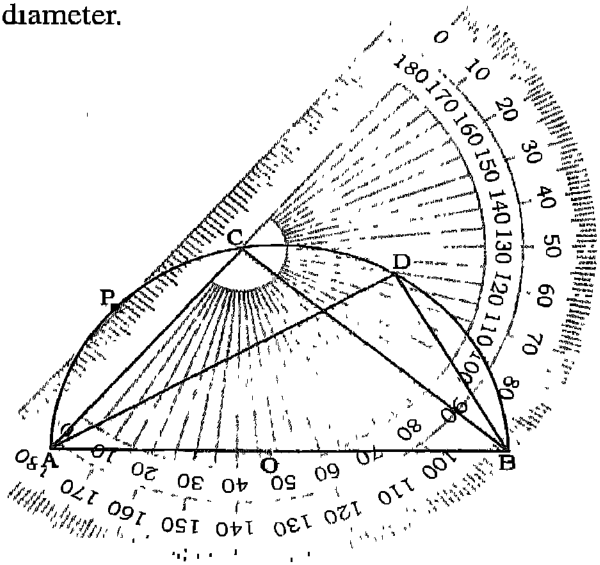


Fig. 13.6

Let us take another point D on the semicircle. Join AD and BD. We get ∠ADB. Measure ∠ADB with the help of a protractor as before. We observe that ∠ADB is also a right angle.

We may use set-squares to measure ∠ACB and ∠ADB (Fig. 13.7). We observe that ∠ACB and ∠ADB are again right angles.

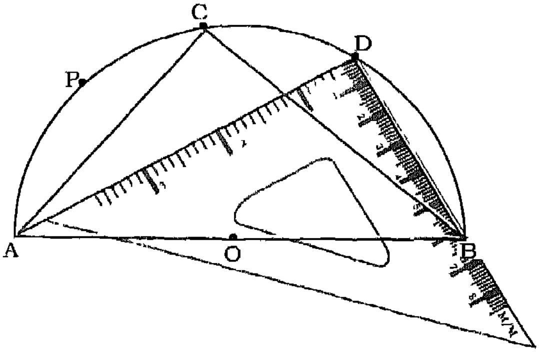


Fig. 13.7

Take a few more points and repeat the above activity. What do you observe? In all cases, we have :

Property I : *Angle in a semicircle is a right angle.*

13.5.1 Angles in a Segment

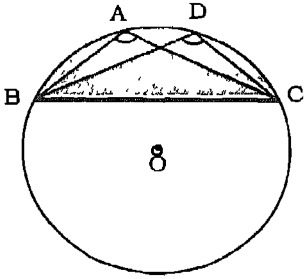


Fig. 13.8

In an earlier section, we learnt to draw angles in a segment. Let us use that method and draw ∠BAC and ∠BDC in the same segment BADC of a circle with centre O (Fig. 13.8).

Measure ∠BAC and ∠BDC using a protractor. Find ∠BAC–∠BDC.

Draw two more circles like Fig. 13.8 and number these circles 1, 2 and 3. Repeat the above procedure for each circle. Record the results in the form of the following table :

Circle	$\angle BAC$	$\angle BDC$	$\angle BAC - \angle BDC$
1.			
2.			
3.			

We observe that $\angle BAC - \angle BDC$, in each case, is equal to zero or so small that we can take it to be zero. Thus we have

$$\angle BAC = \angle BDC$$

In other words, angles in the same segment are equal

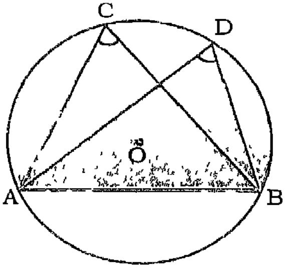


Fig. 13 9

Activity : Draw a circle with centre O (Fig.13.9). Take two points A and B on the circle and join them The chord AB divides the circular region in two parts (segments). Draw $\angle ACB$ and $\angle ADB$ in the same segment as shown in Fig.13.9.
Now, take a sheet of tracing paper and make a trace copy of $\angle ACB$. Superpose the trace copy of $\angle ACB$ on $\angle ADB$, so that C falls on D and CA falls along DA What do you observe? Does CB fall along DB?
Yes, i.e., $\angle ACB$ superposes $\angle ADB$ completely. Thus, $\angle ACD = \angle ADB$. This illustrates the following property:

Property II : *Angles formed in the same segment are equal.*

Example 1 : In segment ADCB of the adjoining figure, $\angle ACB = 66°$. What is the measure of $\angle ADB$?

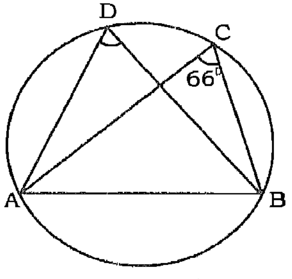


Fig. 13.10

Solution : Using Property II that angles formed in the same segment are equal, we have

$$\angle ADB = \angle ACB = 66°$$

Hence, $\angle ADB = 66°$.

EXERCISE 13.1

1. Draw a circle with centre O and radius 4 cm. Explain the steps necessary to draw a segment of the circle

2. Draw a circle with centre O and any radius. Draw the diameter AB and shade one semicircular region

3. Show four angles in a semicircular region of question 2.

4. In Fig.13.11, $\angle ACB = 60^\circ$. What is $\angle ADB$? Give reason.

5. In a circle of any radius, draw a diameter AB. Take three points C,D, E on one of the semicircle. Draw $\angle ACB$, $\angle ADB$ and $\angle AEB$. Measure each of them with a protractor. Write your observations.

6. In Fig 13.12, find the measures of $\angle APD$ and $\angle ACD$ formed in the segment APCD?

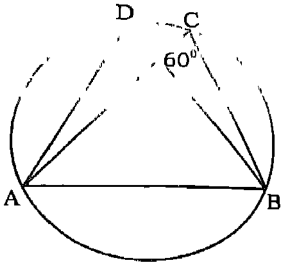


Fig 13.11

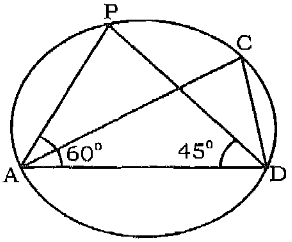


Fig.13.12

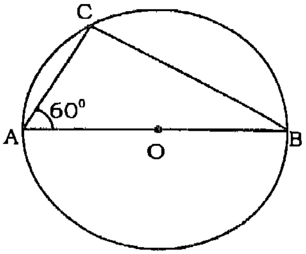


Fig 13 13

7. In Fig.13.13, find the measures of angles ACB and ABC, O being the centre of the circle.

8. Observe Fig.13.14, and find the measures of $\angle RQS$ and $\angle RPS$.

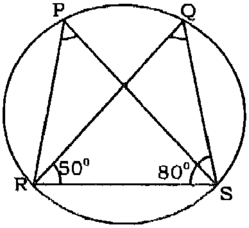


Fig 13.14

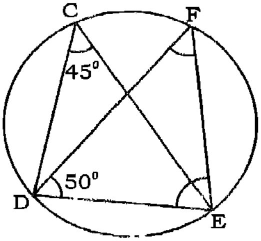


Fig.13.15

9. Find the measures of $\angle DFE$ and $\angle FED$ of Fig.13.15.

10. In Fig 13 16, AB = AC and $\angle ABC = 55^\circ$. Find $\angle BDC$.

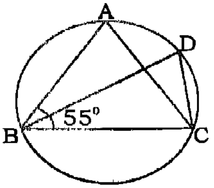


Fig.13 16

Things to Remember

1. The region bounded by an arc of the circle and the chord of the arc is called a segment of the circle.
2. The part of the plane that consists of a diameter, semicircle and interior of the circle that is enclosed by the semicircle and the diameter is called a semicircular region.
3. Angle in a semicircle is a right angle.
4. Angles formed in the same segment are equal

As History Tells Us

The word 'geometry' is said to be derived from the Greek words 'geo' meaning 'Earth' and 'metron' meaning 'measurement' Thus, the origin of geometry can be traced back to the times when people first felt the need of measuring lands.

Ancient Egyptians were perhaps the first people to study geometry in the process of restoring the landmarks affected by the annual flood caused by the river Nile along its bank. However, they were mainly concerned with finding the areas of rectilinear figures such as rectangles, triangles, etc. They also developed geometry for buiding their miraculous pyramids In fact, they were so perfect in their measurements that the relative error in the sides of the square base of the great pyramid is less than $\frac{1}{14000}$. Also the relative error in the corner right angles is $\frac{1}{27000}$.

Babylonians also used geometry only for finding areas of rectilinear figures and developed a number of formulae for areas of these figures. These formulae are available in an ancient Babylonian mathematical text *Rhind Papyrus* (1650 BC). The credit for the degree measure of an angle also goes to Babylonians.

The knowledge of geometry passed on from Egyptians to Greeks who gave it a systematic treatment and extended its study from measurement of areas, etc. (i.e., mensuration) to the study of figures formed by points, lines and planes. In this connection, the name of *Thales* (640-546 BC), a merchant of the city of Miletus, might be mentioned. He had accumulated much wealth in his early years and spent his later years in travelling and studying. While travelling in Egypt, he became interested in geometry and on his return to Greece, he started teaching geometry to his friends. It is believed that the ASA congruence condition was used by *Thales* in finding the distance between the ships in sea

The most famous pupil of *Thales* was *Pythagoras* (580-500 BC) He is believed to have been the first to give a proof of Pythagoras Theorem. The best known of the Greek mathematicians is *Euclid*. The credit for systematically putting together whatever was known till then in mathematics in general and in geometry in particular, goes to *Euclid* His works are in the form of thirteen volumes called the *Elements*. With great clarity and thoroughness, he discusses straight lines, parallelograms, rectangles, squares, circles, tangents and chords. The presentation from enunciation, statement, construction, proof to conclusion was given by *Euclid* and is still followed more or less.

Even though the credit for developing geometry systematically as a science goes to Greeks, geometry was also studied by ancient Indians It is generally believed that the study of geometry started in India in connection with construction of different types of *Vedies* (altars/वेदियाँ) for performing rituals such as '*yajnas*' (यज्ञ). The *vedies* were used to be constructed in several regular shapes. The construction of these needed a lot of knowledge of geometric figures. The *Sulbasutras* (शुल्ब–सूत्र) which are believed to have been composed between 800-500 BC contain formulae for the construction of such *vedies*. The excavations of *Mohenjodaro* and *Harappa* (Now in Pakistan) and *Lothal* (in Gujarat) further reveal that ancient Indians were not only using the knowledge of geometry to construct *vedies* but were using it also for planning the layout of colonies and construction of houses, roads and other buildings. Some of the names of ancient Indian mathematicians who contributed to geometry are *Baudhayana* (800 BC), *Aryabhata* (born 476 AD), *Brahmagupta* (born 598 AD) and *Bhaskara* (born 1114 AD).

AREAS OF RECTANGULAR PATHS

CHAPTER 14

14.1 Introduction

In Class VI, we have learnt about the concepts of perimeter and area In particular, we studied perimeters and areas of rectangles and squares. You may recall the following formulae :

Perimeter of a Rectangle = 2 (Length + Breadth)

Area of a Rectangle = Length ×Breadth

Perimeter of a Square = 4 ×Side

and Area of a Square = $(\text{Side})^2$

If we look around our environment, we come across a number of objects having rectangular or square shapes, e.g , rectangular parks or gardens, floor of a room, pictures, posters, etc. You might have observed that quite often, some space in the form of path is left inside or outside and also in between as cross paths in the parks to enable the people to have their morning or evening walks. Many times, while framing a picture (painting or poster or a photograph), some space (or margin or strip) is left around the picture. Similarly, sometimes we want to build a verandah around a room of our house. In order to plan above type of activities properly, we would like to know the estimated cost for performing these activities, which in general involve rectangular and square paths. In order to do so, we shall learn to calculate areas of rectangular paths in the present Chapter. We shall then apply this knowledge in solving some daily life problems.

14.2 Rectangular Paths

Rectangular paths are generally found around (outside or inside) a rectangle or in the form of central paths. We shall explain the method of calculation of their areas through some examples.

Example 1 : A rectangular lawn of length 40m and breadth 25m is to be surrounded externally by a path which is 2m wide. Find the area of the path.

Solution : Let us represent the lawn by rectangle ABCD and the path around it by shaded region as shown in Fig.14.1.

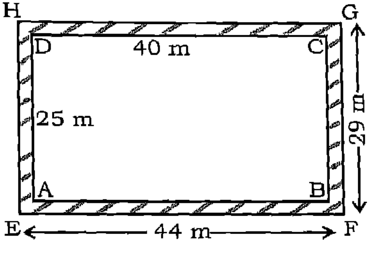


Fig 14.1

Clearly, in this case,

EF = (40 + 2 + 2) m = 44 m

and FG = (25 + 2 + 2) m = 29m

Now, area of the path = Area of rectangle EFGH – Area of rectangle ABCD

$= (44 \times 29 - 40 \times 25)\ m^2$

$= (1276 - 1000)\ m^2$

$= 276\ m^2$

Thus, area of the path is $276\ m^2$

Example 2 : A path 2.5m wide runs inside along the boundary of a square field whose side is 65m. Find the area of the path. Find the cost of manuring the rest of the field at the rate of Rs 5 per m^2.

Solution : Let us represent the square field by square ABCD (Fig.14.2) and the path inside around it by the shaded portion.

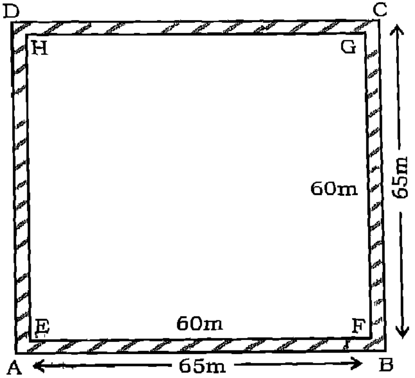


Fig.14.2

Now, area of the path = Area of square ABCD – Area of square EFGH

$= (65 \times 65 - 60 \times 60) m^2$

$= (4225 - 3600) m^2$

$= 625\ m^2$

Also area of the portion to be manured

$= 60 \times 60\ m^2$

$= 3600\ m^2$

$\therefore$ Cost of manuring at the rate of Rs 5 per m^2 = Rs 3600 × 5 = Rs 18000.

Example 3 : Two cross roads, each 2m wide, run at right angles through the centre of a rectangular park of length 72m and breadth 48m such that each is parallel to one of the sides of the rectangle Find the area of the roads. Also, find the area of the remaining portion of the park.

Solution : In Fig.14.3, ABCD represents the park and rectangles PQRS and EFGH represent the roads.

Area of the roads = Area of PQRS + Area of EFGH – Area of KLMN

$= (48 \times 2 + 72 \times 2 - 2 \times 2)\text{m}^2$

$= (96 + 144 - 4)\ \text{m}^2$

$= 236\text{m}^2$

Further, area of the remaining portion of the park

= Area of rectangle ABCD – Area of roads

$= (72 \times 48 - 236)\text{m}^2$

$= (3456 - 236)\text{m}^2$

$= 3220\text{m}^2$

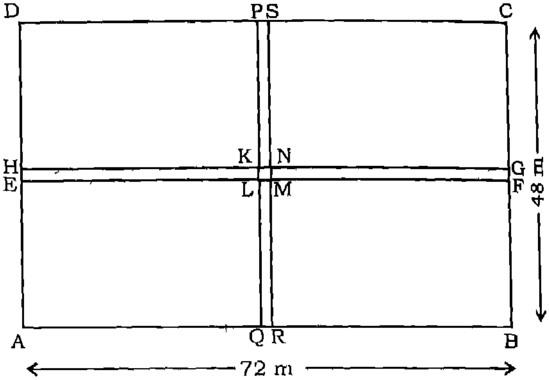


Fig.14.3

EXERCISE 14.1

1. A garden is 90m long and 75m broad. A path 5m wide is to be built outside around it. Find the area of the path.
2. A path 5m wide runs along inside a square park of side 100m. Find the area of the path.
3. A 3m wide path runs outside around a rectangular park of dimensions 125m by 65m (or 125m × 65m). Find the area of the path.

   [**Note :** Many a time, we use the expression 125m by 65m or 125m × 65m to mean that length = 125m and breadth = 65m.]
4. A painting is painted on a cardboard 8cm long and 5cm wide such that there is a margin of 1.5cm along each of its sides. Find the total area of the margin.
5. The side of a square flower bed is 1m 80cm It is enlarged by digging a strip 20cm wide all around it. Find
   - (i) the area of the enlarged flower bed.
   - (ii) the increase in area of the flower bed.

6. A verandah 1.25m wide is constructed all along the outside of a room 5.5m long and 4m wide Find

(i) the area of the verandah.

(ii) the cost of cementing the floor of the verandah at the rate of Rs 25 per m^2

7. A path 1m wide is built along the border inside a square garden of side 30m. Find

(i) the area of the path.

(ii) the cost of planting grass in the remaining portion of the garden at the rate of Rs 2.40 per m^2.

8. A poster of size 20cm by 16cm is pasted on a sheet of cardboard such that there is a margin of 3.5cm along each side of the poster. Find

(i) the total area of the margin.

(ii) the cost of the cardboard used at the rate of Rs1.20 per cm^2

9. Two cross roads, each of width 5m, run at right angles through the centre of a rectangular park of length 700m and breadth 300m parallel to its sides Find the area of the roads. Also, find the cost of constructing the roads at the rate of Rs 105 per m^2.

10. A rectangular garden is 65m long and 50m wide. Two cross paths each 2m wide are to be constructed parallel to the sides. If these paths pass through the centre of the garden, find the cost of constructing the paths at the rate of Rs 69 per m^2.

11. A rectangular field is of dimensions 25m × 16 4m Two paths run parallel to the sides of the rectangle through the centre of the field. The width of the longer path is 1 7m and that of the shorter path is 2m. Find

(i) the area of the paths

(ii) the area of the remaining portion of the field.

12. Through a rectangular field of dimensions 90m × 60m, two roads are constructed which are parallel to the two sides and cut each other at right angles through the centre of the field. If width of each road is 3m, find

(i) the area covered by the roads.

(ii) the cost of constructing the roads at the rate of Rs 110 per m^2

13. There is a rectangular field of length 94m and breadth 32m. Three roads each of width 2m pass through the field such that two roads are parallel to the breadth and the third is parallel to the length. Find the area of the field

(i) covered by the three roads.

(ii) not covered by the roads.

14. A rectangular park is of dimensions 90m by 80m. Four paths pass through the park such that two paths each of width 1.5m are parallel to the breadth and two paths each of width 2m are parallel to the length. Find

(i) area of the paths.

(ii) area of the remaining portion of the park.

Things to Remember

1. Area of a rectangle = length × breadth.

2. Area of a square = $(\text{side})^2$.

3. Area of a rectangular path inside (or outside) a rectangular field = Area of the outer rectangle – Area of the inner rectangle

4. Area of cross paths = Area of all the rectangles making the paths – Area of the common rectangles (or squares).

SURFACE AREAS AND VOLUMES

CHAPTER 15

15.1 Introduction

In Class VI, we have studied the concepts of *perimeter* and *area*. We have also learnt to find areas of some simple plane figures like rectangles and squares. In the previous Chapter, we have learnt about areas of rectangular paths. We will now study some figures that are not plane figures and apply this knowledge in solving some simple daily life problems. The simplest of these figures are *cuboids* and *cubes*. These figures do not lie wholly in a plane. Such figures are called *solid (three dimensional) figures*. We shall learn to find the surface area and volume of a cuboid and also of a cube.

15.2 Cuboid and Cube

Look at the following objects which we come across in our daily life :

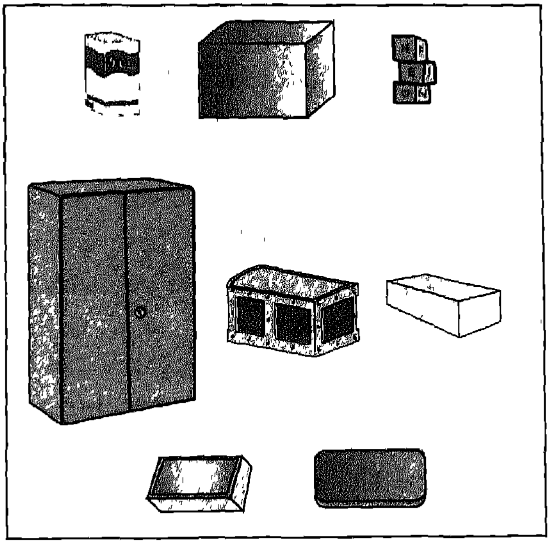


Fig.15.1

All the above objects (which have length, breadth and height) are physical examples of a geometric figure called a *cuboid* (also called a *rectangular parallelopiped*). Fig.15.2 represents a cuboid. It is not a plane figure although it has been drawn on a sheet of paper. It is, in fact, a sketch of a cuboid.

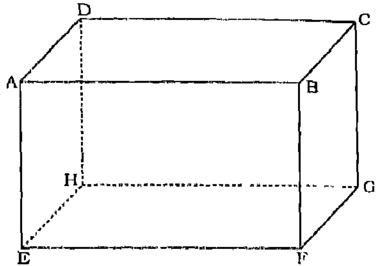


Fig.15.2

As can be seen from the figure, a cuboid is made up of *six* rectangular regions. Each region is called a *face* of the cuboid. Thus, *a cuboid has six faces*. The top and the bottom faces form one pair of *opposite faces*, the front and the back faces form another pair of *opposite faces* and the two side faces form the third pair of *opposite faces*. Note that each face in the above three pairs of *opposite faces is congruent* to the other face. Thus, there are *three pairs of congruent opposite faces* in a cuboid.

In Fig.15.2, ABCD is the top face and EFGH is the bottom face (*or base*). ABFE is the front face and CDHG is the back face, while BFGC and AEHD are the side faces. The faces other than the bottom and top are called *lateral faces also*.

Any two faces, other than the opposite faces, are called *adjacent faces* and these meet in a line segment called an *edge* of the cuboid. In Fig. 15.2, the edges of the cuboid are AB, BC, CD, DA, EF, FG, GH, HE, AE, BF, CG and DH. Thus, *a cuboid has twelve edges*. Note that except for the opposite face all the other four faces are adjacent faces of a given face. Thus, each face has four adjacent faces

In a cuboid, we find eight corners. Each of these is called a *vertex* of the cuboid. In Fig. 15.2, A, B, C, D, E, F, G and H are the vertices of the cuboid. At each vertex, three edges meet. Note that, of these three edges, each edge is perpendicular to the other two.

Since the opposite faces of a cuboid are congruent, it is easy to observe from the figure that

$$AB = DC = HG = EF$$

$$BC = AD = FG = EH$$

and

$$AE = BF = CG = DH$$

Thus, we observe that the twelve edges of a cuboid can have only *three* distinct lengths. Usually, the longest of these is called the *length* of the cuboid and out of remaining two, one is called the *breadth* (*or width*) and the other the *height* (*or depth or thickness*) of the cuboid Generally, the length of the vertical sides is taken as the height of the cuboid.

The length, breadth and height of a cuboid are usually denoted by the letter symbols l, b and h respectively. Length, breadth and height of a cuboid are also called the *three dimensions* of the cuboid.

It can be easily seen from the figure that the height of the cuboid with base EFGH is AE (or BF or CG or DH).

A cuboid whose length, breadth and height are equal is called a *cube* (Fig.15.3).

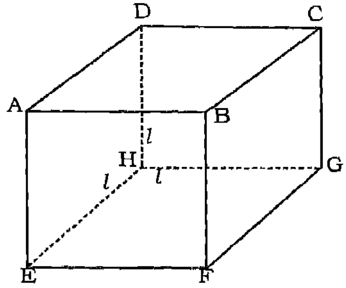


Fig.15.3

Remark : The above description of the cuboid brings to our mind two distinct but related figures: the *hollow cuboid* and the *solid cuboid*. In fact, *by a cuboid we actually mean a hollow cuboid.* It is the figure in space formed by three pairs of opposite faces (rectangular regions) such that whenever two faces (other than opposite faces) meet, they meet in a line segment.

The part of the space enclosed by a cuboid is called its *interior*. A cuboid along with its interior is called a *cuboidal region*, which is generally referred to as a *solid cuboid*. Recall the difference between a rectangle and a rectangular region from earlier classes. Do you see a similar difference between a cuboid (hollow cuboid) and a cuboidal region (solid cuboid)?

In common usage, the word 'cuboid' is used for both—the hollow cuboid and the solid cuboid. In practice, it should not cause any problem, for the context in which it is used, will make the meaning clear. Like cuboid, the word 'cube' is also used for both—the hollow cube and the solid cube (or cubical region).

EXERCISE 15.1

1. Name any four objects from your environment having the shape of a

   (i) Cuboid (ii) Cube

2. Draw a diagram to represent a cuboid, label its vertices as P, Q, R, S, T, U, V and W. Now write the names of its

   (i) faces (ii) edges

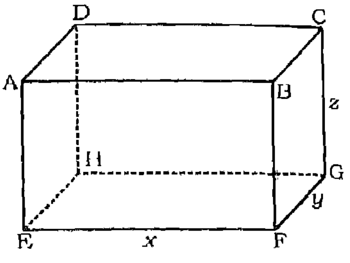


Fig. 15.4

3. Fig.15.4 represents a cuboid. The length of some of the edges are marked by the symbols x, y and z. Indicate the lengths of all the other edges
4. In Fig.15.4. if EFGH be called the base, then name the four side faces
5. In Fig.15.4, if EFGH be the base, name a line segment representing the height of the cuboid.
6. In Fig.15.4, name the
   (i) face opposite to AEHD.
   (ii) faces adjacent to BFGC.
   (iii) faces which meet in the edge AB.
   (iv) three edges which meet in the vertex H.
7. In Fig.15.4, name the three faces which have vertex A in common. Also, name the vertex, in which the other three faces meet. Is this vertex G? *Vertices* A *and* G *are called a pair of opposite vertices The line segment* AG *is called a diagonal of the cuboid.* How many diagonals are there in the cuboid? Name all of them.
8. Trace the diagram given in Fig. 15.5 on a tracing paper. Make a copy of it on a stiff paper. Cut it out and fold it along the dotted lines in such a way that the line segments marked with the same number are parallel to each other or coincident. Also, no edge is to coincide with an edge numbered differently. What is the shape of the solid you obtain ? Stick the edges with the help of a cello-tape. Paint the opposite faces of the solid so obtained in same colour.
   [**Remark** : Fig.15.5 is called the *net* of a cuboid.]

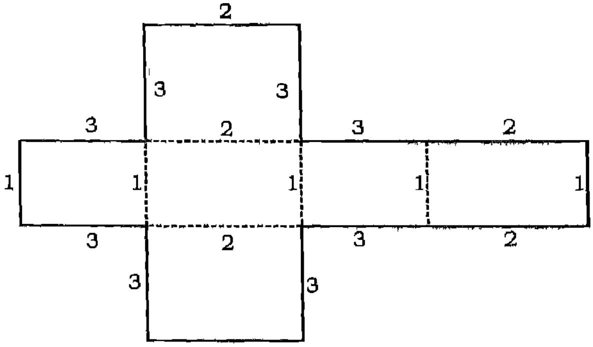


Fig 15.5

9. Trace the diagram given in Fig.15.6 (which is made of six congruent squares) on a tracing paper. Make a copy of it on a stiff paper Cut it out and fold it along the dotted lines as in the case of a cuboid. What is the shape of the solid you obtain? Stick the edges with the help of cello-tape. Paint the opposite faces of the solid so obtained in same colour.

[**Remark** : Fig 15.6 is called the *net* of a cube.]

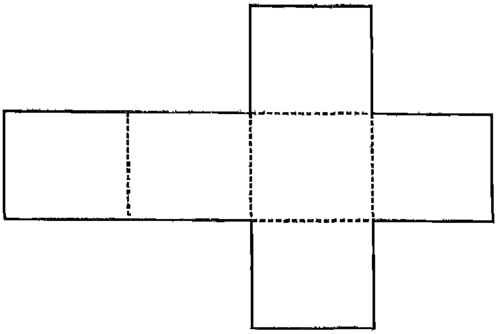


Fig.15 6

10. Fill in the blanks to make the following statements true :

(i) The number of faces of a cuboid is ________

(ii) The number of edges of a cuboid is ______ .

(iii) Two adjacent faces of a cuboid meet in a line segment called its _______

(iv) The length, breadth and height of a cuboid are called its three_______ .

(v) The number of lateral faces of a cuboid is _______ .

(vi) A __________ is a cuboid in which all the edges are equal.

(vii) The number of vertices of a cuboid is _______

(viii) Three concurrent edges of a cuboid meet at a point called a ______ of the cuboid.

(ix) All the faces of a cube are ______ .

(x) A cuboid has ______ pairs of opposite faces.

(xi) Two edges of a cube through a vertex make an angle of _____ with each other.

(xii) A cube has _________ diagonals.

15.3 Surface Area

You might have observed that most of the articles sold in the market are packed in tins, cartons, cardboard or wooden boxes, etc Most of these packings are cuboid in shape It is natural for a manufacturer to find out in advance as to how much tin sheet, cardboard or steel sheet is needed to make the packing boxes, tins, cartons, etc In order to find out this, there is a need to calculate the surface area of a cuboid

We already know that surface of a cuboid consists of six rectangular faces (Fig.15.7). The sum of the areas of these six rectangular faces is called the *total surface area of the cuboid.*

Let the length, breadth and height (in cm) of a cuboid be l, b and h respectively Then, area of bottom (base) and top faces

$= (l \times b + l \times b)$ cm^2

$= 2lb$ cm^2,

area of side faces $= (b \times h + b \times h)$ cm^2

$= 2bh$ cm^2

and area of front and back faces

$= (h \times l + h \times l)$ cm^2

$= 2hl$ cm^2

$\therefore$ total surface area $= (2lb + 2bh + 2hl)$cm^2

$= 2\ (lb + bh + hl)$cm^2

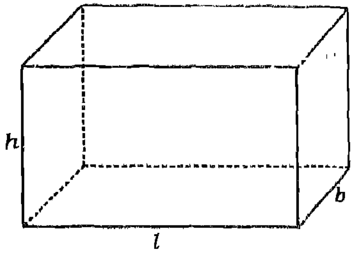


Fig.15.7

Thus, total surface area of a cuboid of length l, breadth b and height l units is $2(lb + bh + hl)$ square units.

Clearly, the lateral surface area of the cuboid $= 2(lh + bh)$ square units

$= 2\ (l + b)\ h$ square units

= perimeter of the base × height

Note that this formula can also be used for finding the area of the four walls of a rectangular room.

In the case of a cube, $l = b = h$.

Therefore, surface area of a cube of side l units

$= 2\ (l \times l + l \times l + l \times l)$ square units

$= 6\ l^2$ square units

and its lateral surface area $= 2\ (l \times l + l \times l) = 4\ l^2$ square units.

Remarks : 1. Length, breadth and height must be expressed in the same units, for calculation of surface area.

2. For 'total surface area', the phrase *whole surface area* or simply the *whole surface* is also used. Similarly, for lateral surface area, we may use sometimes the term lateral surface.

Let us illustrate the use of these formulae through some examples.

Example 1 : Find the surface area of a cuboid whose length, breadth and height are respectively 10cm, 8cm and 5cm

Solution : Here, l = 10cm, b = 8cm and h = 5cm.

$\therefore$ surface area of the cuboid $= 2\,(lb + bh + hl)$

$$= 2\,(10 \times 8 + 8 \times 5 + 5 \times 10)\text{cm}^2$$
$$= 2 \times 170\text{cm}^2 = 340\text{cm}^2$$

Example 2 : The edge of a cube is of length 5m. Find the surface area of the cube.

Solution : Here, l = 5m.

Now, surface area of a cube $= 6l^2$

$$= 6 \times (5)^2\,\text{m}^2$$
$$= 6 \times 25\text{m}^2$$
$$= 150\text{m}^2$$

Example 3 : A plastic box 1.5m long, 1.25m wide and 65cm deep is to be made. It is to be open at the top. Ignoring the thickness of the sheet, determine the

(i) area of the sheet required to make the box,

(ii) cost of sheet for it, if a sheet measuring 7m × 3.5m costs Rs 510.

Solution : (i) Here, l = 1.5m, b = 1.25m and h = 65cm $= \frac{65}{100}\text{m} = 0.65\text{m}$

As it is an open box, surface area

= area of bottom face + area of lateral faces

$$= l \times b + 2\,(l + b) \times h$$
$$= 1.5 \times 1.25\text{m}^2 + 2\,(1.5 + 1.25) \times 0.65\text{m}^2$$
$$= 5.45\text{m}^2$$

(ii) Area of the sheet $= 7 \times 3.5\text{m}^2$

$$= 24.5\text{m}^2$$

Now cost of 24.5m² sheet is Rs 510,

$\therefore$ cost of 1m^2 sheet is Rs $\dfrac{510}{24.5}$

$\therefore$ cost of 5.45m^2 sheet = Rs $\dfrac{510 \times 5.45}{24.5}$

= Rs 113.45

Example 4 : The length, breadth and height of a room are 5m, 4m and 3m respectively. Find the cost of white-washing the walls of the room and the ceiling at the rate of Rs 7.50 per m^2.

Solution : Area of the four walls $= 2\,(l + b) \times h$

$= 2\,(5 + 4) \times 3\ \text{m}^2$

$= 54\text{m}^2$

Area of the ceiling $= l \times b = 5 \times 4\text{m}^2 = 20\text{m}^2$

$\therefore$ Total area to be white-washed $= (54 + 20)\ \text{m}^2$

$= 74\text{m}^2$

Therefore, cost of white-washing at the rate of Rs 7.50 per m^2

= Rs 7.50×74

= Rs 555

Remark : In all the questions involving four walls of a hall or a room, we shall ignore the presence of windows or doors in the walls.

EXERCISE 15.2

1. Give 3 examples from our environment where finding of surface area may be required.
2. Find the total surface area of a lunch box with dimensions 15cm, 9cm and 8cm.
3. Find the surface area of a chalk box whose length, breadth and height are 16cm, 8cm and 6cm respectively.
4. Find the surface area of a closed cardboard box of length 0.5m, breadth 25cm and height 15cm.
5. Find the surface area of a cube with edge
   (i) 11cm (ii) 1.2m (iii) 27cm
6. Find the surface area of a wooden box whose shape is of a cube of edge 15cm.
7. The dimensions of an oil tin are 26cm × 26cm × 45cm Find the
   (i) area of the tin sheet required to make 20 such tins
   (ii) cost of the tin sheet for making these tins at the rate of Rs 20 per m^2.

8. A box with lid is in the form of a cuboid. Its length, breadth and height are respectively 1m 30cm, 75cm and 20cm. Find the cost of painting its outside at the rate of Rs 4 per cm^2.

9. The length, breadth and height of a cuboidal box are respectively 2m 10cm, 1m and 80cm. Find the
   (i) area of the canvas required to cover this box
   (ii) cost of the canvas for covering the box at the rate of Rs 50 per m^2

10. A barn with a flat roof is rectangular in shape of breadth 10m, length 15m and height 5m. It is to be painted inside on the walls and on the ceiling but not on the floor. Find the total area to be painted

11. The walls and ceiling of a room are to be plastered with cement. If the length, breadth and height of the room are respectively 4.5m, 3m and 3.5m, find the area to be plastered.

12. A classroom is 11m long, 8m wide and 5m high. Find the sum of areas of its floor and the four walls.

13. A swimming pool is 20m in length, 15m in breadth and 4m in depth. Find the cost of cementing its floor and walls at the rate of Rs 36 per m^2

14. The floor of a rectangular hall has a perimeter 250m. If its height is 6m, find the cost of painting its four walls at the rate of Rs 20 per m^2.

15. The paint in a certain container is sufficient to paint an area equal to $9.375m^2$. How many bricks of dimensions 22 5cm × 10cm × 7.5cm can be painted out of this container? [*Hint:* Find the surface area of a brick.]

15.4 Volume

Just as in the case of plane regions, given two solid (space or three dimensional) regions, we can talk of one being bigger or larger than the other. Thus, two solid regions can be compared. In other words, a solid region has a *magnitude or size or measure*. The measure or magnitude of a solid region is called its *volume*. In other words, the measure of the space occupied by a solid (in fact solid region) is called its volume. Volume has great practical utility For example, consider the following real life problems :

1. A rectangular overhead tank is built for storage of water. The larger the volume of the cuboidal region, the greater the quantity of water that can be stored.
2. A rectangular tin box is to be made to store oil. The greater the volume of the cuboidal region, the more is the quantity of oil it can store.

3. A school wall is to be built and we want to know the number of bricks required for the purpose. Now a wall is a cuboidal region, and so is a brick. Knowing the volume of a brick and that of the wall, we can find the number of bricks

Remark : Note that volume is the magnitude of a cuboidal region However, we often speak of the volume of a cuboid, meaning thereby the volume of the cuboidal region determined by it.

Many a time, it is possible to decide by observation alone as to which of the two solid regions is bigger or smaller. For example,

(i) The volume of an inflated football lying in a box is much smaller than the volume of the box.

(ii) The volume of a box placed in a room is much smaller than the volume of air in the room.

(iii) The volume of a suitcase is much smaller than the volume of an almirah.

(iv) The volume of a railway wagon is much larger (or bigger) than the volume of an almirah.

However, sometimes, observation may fail to guide us. Look at the regions A and B given in Fig.15.8. It is true that A and B contain the same amount of milk Yet region A appears to be larger than the region B. Thus, while observation often gives us a quick result, the result may turn out to be incorrect. Therefore, we need a better method of comparing space regions. For this, we adopt the same method of measuring volumes as adopted for measuring areas and lengths in earlier classes.

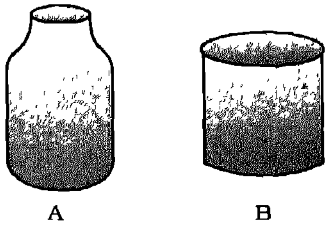


Fig.15.8

15.5 Measurement of Volume – Need for a Standard Unit

Let us compare the two regions A and B given in Fig.15.9. Obviously, region A is larger than B. How many times larger? To answer this question, let us start with the empty drum A and pour water into it with the help of the bucket B. Suppose 20 buckets full of water, when poured into the drum, fill it up completely. Then, we say that

$$\text{Volume of A} = 20 \times (\text{Volume of B})$$

If we agree to call B a unit, then we may say

Volume of A = 20 units (B = 1 unit of volume)

We have measured region A above in terms of region B. Region A may contain a fractional number (instead of 20) of region B. In that case, volume of A will be that fractional number. Further, region A can be measured by some other person in terms of, say region C (Fig.15.10). Obviously, in such a case, the volume of region A will not be 20 units but it may be about 40 units (C = 1 unit of volume)

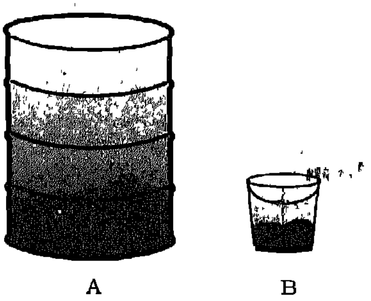


Fig.15.9

Fig 15.10

Thus, if everybody chooses his/her own unit, the volume obtained by one person may not be the same as obtained by another. For this reason, it is necessary to have a common *standard unit of volume* which can be understood by everyone.

15.6 Some Standard Units of Volume

Recall that a square of side 1cm, 1mm or 1m was taken as a standard unit for measuring areas in earlier classes and areas were expressed as square centimetres (cm^2) or square millimetres (mm^2) or square metres (m^2), etc. In a similar manner, we take a cube of side 1cm (or 1mm or 1m) as a standard unit for measuring volumes (Fig.15.11) and express the volume as cubic centimetres (cm^3) or cubic millimetres (mm^3) or cubic metres (m^3). If a solid region S contains V such unit cubes, we say that

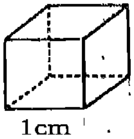
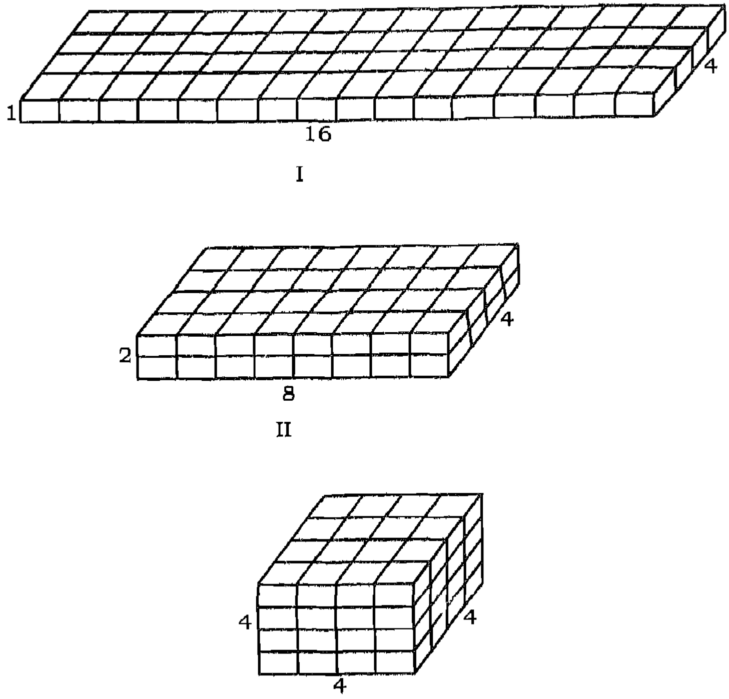


Fig.15 11

$S = V\ cm^3$ or Vm^3 or Vmm^3, depending upon the unit used.

15.7 Volume of a Cuboid and a Cube

You might have played with cubical blocks of wood or plastic materials when you were younger. Let us take 64 equal cubes of say edge 1cm and put these cubes in the form of a cuboid. Obviously, we can make a number of different cuboids with these cubes. Three of these cuboids are shown in Fig.15.12.



I

II

III

Fig.15 12

What can we say about the volume of each of these cuboids? Since each of these is made by putting together 64 unit cubes, the volume of each of these is 64 units of volume (cubic units), i.e., 64 cm^3.

What about the length (l), breadth (b) and height (h) in each of the above cuboids and their product $l \times b \times h$?

We can list our observations in the form a table as shown below :

Cuboid	*Length* (l) *in cm*	*Breadth* (b) *in cm*	*Height* (h) *in cm*	*Volume* ($l \times b \times h$) *in* cm^3
I	16	4	1	64
II	8	4	2	64
III	4	4	4	64

We can perform this type of activity by taking some other suitable number of unit cubes and record our observations in the form of a table as above.

These activities and observations suggest that volume V of a cuboid of length l, breadth b and height $h = l \times b \times h$

i.e , $V = l \times b \times h$

Note that V is the volume of the cuboid (in cm^3), if l, b and h are in cm. Similarly, for other units.

Remarks : 1. Length, breadth and height must be expressed in the same units.

2. From the above formula, we also observe that

$$Length = \frac{Volume}{Breadth \times Height}$$

$$Breadth = \frac{Volume}{Length \times Height}$$

$$Height = \frac{Volume}{Length \times Breadth}$$

3. In the case of a cube, $l = b = h$. Therefore,

Volume of a cube $= l \times l \times l = l^3$ cubic units

4. As 1cm = 10 mm, therefore

$1cm^3 = 10 \times 10 \times 10\ mm^3 = 1000\ mm^3$

Similarly,

$1m^3 = 100 \times 100 \times 100\ cm^3 = 1000000\ cm^3 = 10^6\ cm^3$

$= 1000 \times 1000 \times 1000\ mm^3 = 10^9 mm^3$

5. For measurement of volumes of liquids, usually we use the terms litres (l) and millilitres (ml). Further,

$1cm^3 = 1ml$,

$1000cm^3 = 1\ l$

and $1m^3 = 1000000cm^3 = 1000\ l = 1\ kl$ (1 kilolitre)

6. Volume of a container can be expressed in terms of litres, millilitres, etc. In such a case, this volume is called its *capacity*.

We now take some examples to illustrate the use of these formulae.

Example 5 : Find the volume of a block of wood whose length, breadth and height are respectively 10cm, 5cm and 3cm.

Solution : $V = l \times b \times h$

$\therefore \quad V = 10 \times 5 \times 3\text{cm}^3 = 150\text{cm}^3$

Thus, volume of the block is 150cm^3.

Example 6 : Find in litres, the volume (capacity) of a storage tank whose length, breadth and depth are respectively 6.3m, 4.5m and 3.6m.

Solution: Volume of the tank

$$\begin{aligned} &= l \times b \times h \\ &= 6.3 \times 4.5 \times 3.6\text{m}^3 \\ &= 102.06\text{m}^3 \\ &= 102.06 \times 100 \times 100 \times 100 \text{ cm}^3 \\ &= 102060000 \text{ cm}^3 \\ &= 102060 \text{ litres} \quad (\because 1000 \text{ cm}^3 = 1 \text{ litre}) \end{aligned}$$

Thus, the capacity of the tank is 102060 litres.

Example 7 : Find the volume of a cube whose edge is 8m.

Solution : Volume

$$\begin{aligned} &= l^3 \\ &= 8 \times 8 \times 8\text{m}^3 \\ &= 512 \text{ m}^3 \end{aligned}$$

Example 8 : Capacity of a tank is 60 *kl*. If the length and breadth of the tank are respectively 5 m and 4 m, find its depth.

Solution : Volume of the tank = 60 *kl*

$$\begin{aligned} &= 60000\, l \\ &= 60\text{m}^3 \qquad (\because 1000\, l = 1\text{m}^3) \end{aligned}$$

$$\therefore \quad \text{Depth} = \frac{\text{Volume}}{\text{length} \times \text{breadth}}$$

$$= \frac{60}{5 \times 4}\text{m} = 3\text{m}$$

Thus, the depth of the tank is 3m.

EXERCISE 15.3

1. Find the volume of the cuboid whose length, breadth and height are respectively :
   (i) 12cm, 10cm, 8cm (ii) 16.5cm, 8.4cm, 12.7cm
   (iii) 121mm, 54mm, 256mm (iv) 18.5m, 8.3m, 2.9m
   (v) 8m, 70cm, 90cm (vi) 1.5m, 25cm, 15cm
2. Find the volume of a cube whose edge is
   (i) 15mm (ii) 12.5cm
   (iii) 2.6m (iv) 1.72m
3. A cuboidal solid wooden block contains 36cm^3 wood. If its length and breadth are respectively 4cm and 3cm, find its height.
4. A match box measures 4cm by 2.5cm by 1.5cm. What will be the volume of a packet containing a maximum of 12 such match boxes?
5. A cuboidal water tank is 6m long, 5m wide and 4.5m deep. How many litres of water can it hold?
6. A cuboidal vessel is 10m long and 8m wide How high must it be made to hold 480 cubic metres of a liquid?
7. Find the length of a wooden plank of width 2.5m, thickness 0.025m and volume 0.25m^3
8. Find the cost of digging a cuboidal pit 8m long, 6m broad and 3m deep at the rate of Rs 30 per m^3.
9. The capacity of a cuboidal tank is 50000 l of water. Find the breadth of the tank, if its length and depth are respectively 2.5m and 10m.
10. A village, having a population of 4000, requires 150 l water per head per day. It has a tank measuring 20m by 15m by 6m. For how many days the water of this tank will last?
11. Two cubes, each of side 6cm, are joined end to end. Find the volume of the resulting cuboid
12. A godown measures 40m × 25m × 15m. Find the maximum number of wooden crates each measuring 1.5m × 1.25m × 0.5m that can be stored in the godown.

    [*Hint:* Numer of wooden crates = $\dfrac{\text{volume of godown}}{\text{volume of a crate}}$.]
13. How many wooden cubical blocks of edge 20cm can be cut from a log of wood of size 8m by 5m by 80 cm, assuming there is no wastage?
14. How many wooden cubical blocks of edge 12cm can be cut from another cubical block of wood of edge 3m 60cm ?
15. What will happen to volume of a cube, if its each edge is
    (i) doubled? (ii) halved? (iii) tripled?

Things to Remember

1. A cuboid is a figure in space, formed by three pairs of opposite congruent rectangular faces, such that whenever two faces meet, they meet in a line segment.

2. A cuboid has 6 faces, 12 edges and 8 vertices.

3. The part of the space enclosed by a cuboid is called its interior.

4. A cuboid along with its interior is called a cuboidal region.

5. The length, breadth and height of a cuboid are called its three dimensions.

6. A cuboid whose length, breadth and height are all equal is called a cube.

7. The magnitude or measure of a space (solid) region is called its volume

8. A cubic centimetre is the volume of the region formed by a cube of side (or edge) 1 cm.

9. With usual meaning for V, l, b and h

(i) Surface area of a cuboid = $2(lb + bh + hl)$.
(ii) Surface area of a cube = $6l^2$.
(iii) Lateral surface area of a cuboid = $2(l + b)h$.
(iv) Volume of a cuboid (V) = $l \times b \times h$.
(v) Volume of a cube (V) = l^3.

10. Standard units of volume

(i) $1 \text{ m}^3 = 1000000 \text{ cm}^3$
(ii) $1 \text{ cm}^3 = 1000 \text{ mm}^3$.
(iii) $1 \text{ cm}^3 = 1\ ml$.
(iv) $1 \text{ m}^3 = 1000\ l = 1\ kl$.

STATISTICS

CHAPTER 16

16.1 Introduction

The term statistics is derived from the Latin word *status* meaning condition. The information or facts about an object or phenomenon are usually available as a collection of numerical values, which are called *data**. To know more about the phenomenon, we need to analyse the data about it. The methods of analysis of any collection of data are covered by the branch of mathematics called *Statistics*.

It is difficult to draw conclusions from a large amount of numerical data. So, if data are presented pictorially, the same become understandable. We draw many conclusions by just looking at the pictorial representation of data. In this Chapter, we shall learn to represent data pictorially and also reading, interpreting and drawing of bar graphs.

16.2 Bar Graphs

A survey was carried out in a certain school to find about different modes by which students travel to school each day. 30 students of Class VI were interviewed and the data so obtained were recorded in the following table:

Table 1 : Common Modes of Travelling to School

Travelling mode	*Walk*	*Cycle*	*School Bus*	*Public Bus*	*Scooter*
Number of students	7	3	11	5	4

To study the characteristics of the data, it is quite helpful to use pictures. Expressing numerical data through pictures is called *pictorial representation of data*. Let us use one symbol for one student of Table 1 as shown in Fig. 16.1.

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* The word *data* is the plural form of *word datum*.

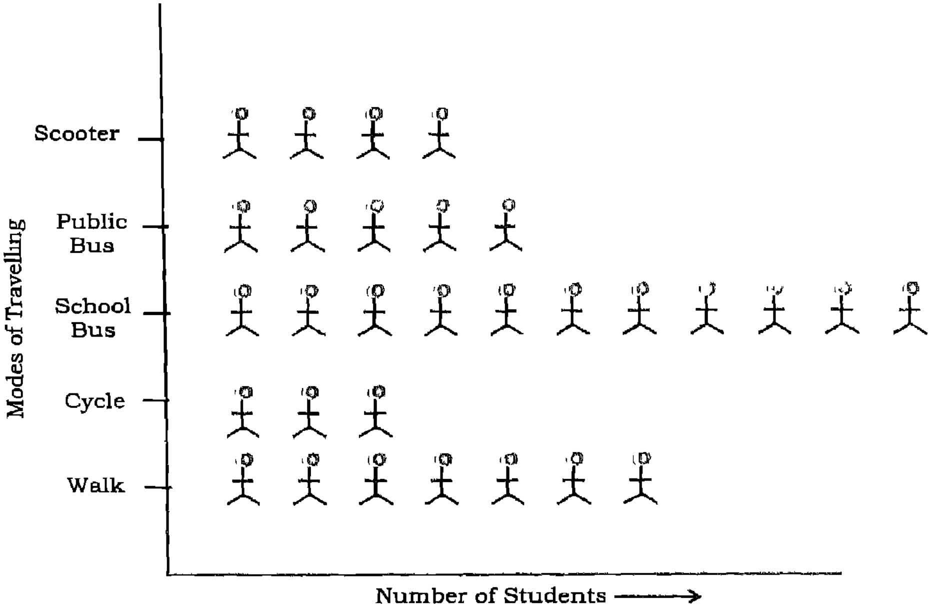


Fig. 16 1

Such a pictorial representation of data is known as *a pictograph*. The above pictograph shows immediately that school bus is the most popular mode of travel. Although, we could have drawn this conclusion from the table directly by comparing the numbers 7, 3, 11, 5, 4. But, it is easier to decide by just *looking at the pictograph*. Moreover, in case of huge data, comparison of numbers may be difficult and time-consuming, whereas to know facts from a pictograph, it is sufficient to just glance at it.

Let us consider another survey, which was carried out in a certain school to find out about the popular school subject among Class VII students. The data giving the number of *votes for each subject* have been recorded in the following Table 2 :

Table 2 : Popular School Subjects among Class VII Students

Subject	*Hindi*	*English*	*Geography*	*Mathematics*	*Science*
Number of students	30	20	21	35	31

We shall again use symbols to represent the given data. Let one symbol of a book represent five votes (*units*) and let some incomplete symbol represent part units (less than five units) (Fig.16.2)

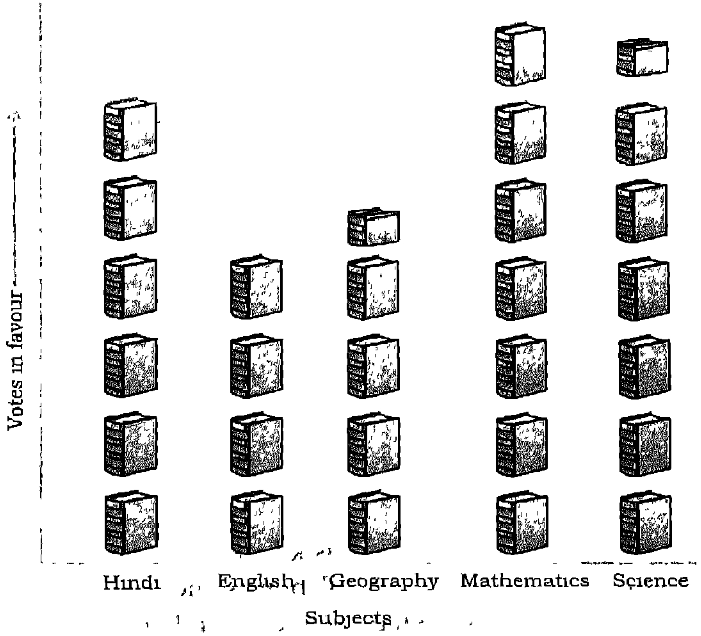


Fig.16.2

What do you observe from Fig.16.2? Mathematics compared to other subjects is the most popular school subject among Class VII students.

We have learnt from the above discussion that representing numerical data by pictographs helps us in finding its special features at a glance. But this way of representing data is not only time-consuming but difficult also, specially when we have to draw picture and part picture symbols. Thus, instead of picture symbols, it is easy to draw *bars* (rectangles) that represent the given data. Bars of uniform width corresponding to given data are erected horizontally or vertically with equal spacing between them.

For a better understanding, let us consider the bar graph of the following problem connected with the vehicular pollution

Example 1: The vehicular traffic at a busy road crossing in Delhi was studied by the traffic police on a particular day. The number of vehicles passing through the crossing every hour from 6 am to 1 pm is recorded in Table 3. The data have been rounded off to tens.

Table 3 : Vehicular Traffic at a Road-crossing (6 am to 1 pm)

Time in hours	6–7	7–8	8–9	9–10	10–11	11–12	12–1
Number of Vehicles	50	650	1650	1250	850	750	550

The bar graph of the information given in Table 3 is shown in Fig.16.3.

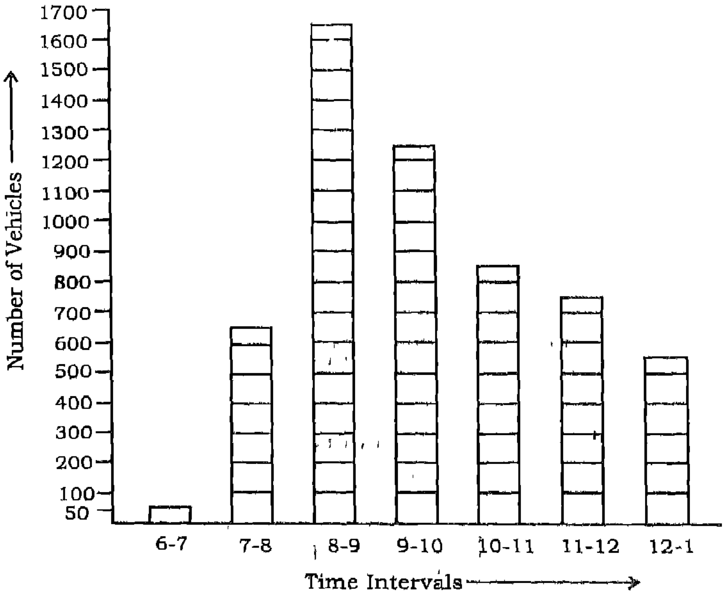


Fig.16.3 The bar graph shows the number of vehicles passing through a crossing during different time intervals.

Note that the given data may also be represented by placing the time intervals vertically and the bars horizontally as shown in Fig.16.4. However, generally vertical bar graphs (Fig 16.3) are preferred.

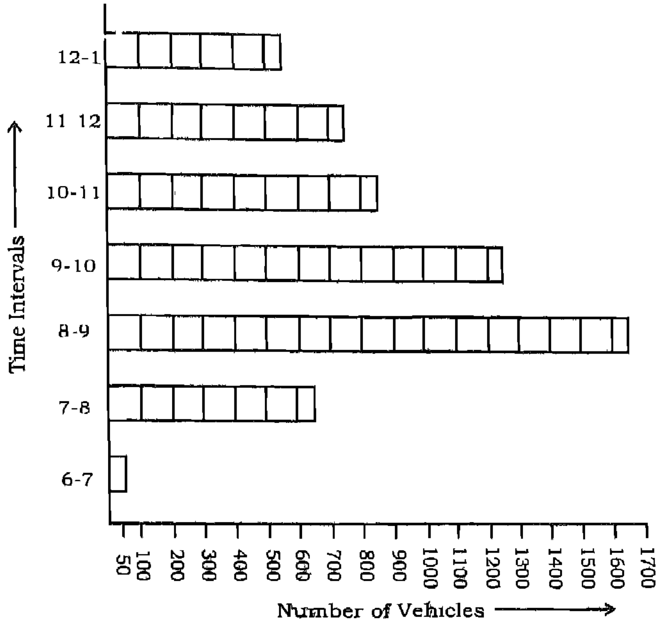


Fig.16.4: The bar graph shows the number of vehicles passing through a crossing during different time intervals.

Thus, we observe that *a bar graph is a pictorial representation of the numerical data by a series of bars or rectangles of uniform width erected horizontally or vertically from the same base line with equal spacing between the bars.* Each rectangle or bar represents only one value of the numerical data. So, there are as many bars as the number of values in the numerical data. The height or length of each bar represents, on a suitable scale, the value of the numerical data. Before we proceed to learn the construction of bar graphs, we shall learn to read and interpret a given bar graph.

16.3 Reading of Bar Graphs

To read a bar graph, we need to observe certain points carefully. For example, by reading the bar graph given in Fig.16.3, we observe the following :

(i) The bar graph shows the number of vehicles passing through a busy road crossing in Delhi from 6 am to 1 pm on a particular day.

(ii) Time intervals are shown on the horizontal line and the number of vehicles along the vertical line, on a suitable scale.

(iii) The scale is 1 unit length equal to 100 vehicles.

(iv) The height of a bar indicates the number of vehicles passing through the crossing during the time interval corresponding to the bar.

16.4 Interpretation of the Bar Graph

By an interpretation of the given bar graph, we mean to draw conclusions from it. Let us consider the bar graph of Fig 16.3. What conclusions can be drawn from it ?

1 The bar graph represents the number of vehicles passing through a particular crossing of Delhi during different time intervals on a particular day.

2. The longest bar corresponds to the maximum traffic. Thus, maximum number of vehicles (1650) passed through the crossing during the hour 8–9.

3. The shortest bar corresponds to the minimum traffic (50) that occurs between the hour 6 – 7.

4 The total traffic during the two peak hours of morning rush (for schools, offices and business establishments) as shown by the two long bars is 1650 + 1250 = 2900 vehicles during the time interval 8–10.

This way, bar graphs help in easy comprehension of given data and conclusions can be drawn at a glance.

Example 2 : Read the bar graph given in Fig. 16.5 and answer the following questions :

(i) What is the information given by the bar graph?

(ii) What is the order of the change in the number of students over several years?

(iii) In which year is the increase in the number of students maximum?

(iv) State whether true or false :

'The number of students during 2000-01 is twice that of 1999-2000'.

Solution : The answers to the questions asked are :

(i) The bar graph represents the number of students in Class VII of a school during the academic years 1997-98 to 2000-01.

(ii) The order of change in the numbers of students is uniform (same) over several years.

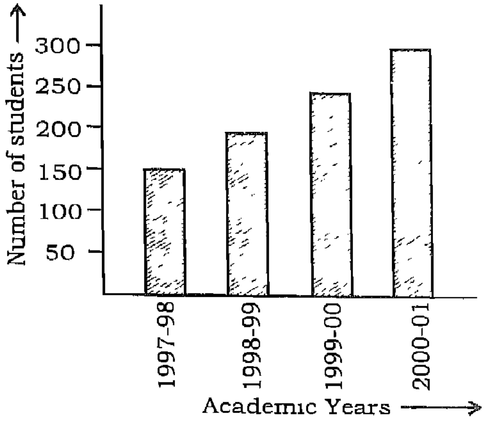


Fig.16 5 · Bar graph of the number of students in Class VII of a school during academic years 1997-98 to 2000-2001

(iii) The increase in the number of students is uniform as the increase in height of the bars is uniform (equal). Hence, in this case, no particular year shows a greater increase than the others.

(iv) The height of the bar corresponding to the year 2000-01 is not double that of the bar corresponding to the year 1999-2000. So, the given statement is false.

16.5 Drawing Bar Graphs on Graph Paper

We now learn to draw a bar graph. We shall use graph paper to construct the bars for ease and accuracy But it is not necessary to do so.

We must keep the following rules in mind for drawing a bar graph :

(i) It should have a title at the top or a caption below the graph to indicate what it is about.

(ii) Scale should also be mentioned below the graph.

(iii) All bars should be of the same width.

(iv) Spaces between the bars should be of the same width.

(v) Each bar should be labelled.

We shall illustrate the construction of bar graphs through examples.

Example 3 : The following data give the amount of loans (in crores of rupees) given out by a bank during some years :

Year	*Loan (in crores of rupees)*
1996	20
1997	25
1998	30
1999	45
2000	60

Construct a bar graph representing the above information.

Solution : We go through the following steps to construct the bar graph:

Step 1 : We take a graph paper and draw two lines perpendicular to each other and call them horizontal and vertical axes(Fig 16.6)

Step 2 : Along the horizontal axis, we mark the information 'years' and along the vertical axis, we mark the corresponding loans in crores of rupees.

Step 3 : According to the given data, we choose along horizontal axis, a uniform (equal) width of the bars and a uniform gap between them suitably.

Step 4 : We choose a suitable scale along the vertical axis in the light of the data given to us. Let us choose the scale 1 unit of graph paper = 10 crores of rupees.

Then we mark the corresponding values according to this scale, on the vertical axis. This gives us the heights of the bars.

Step 5 : We calculate below the heights of the bars for various years :

$$1996 : \frac{1}{10} \times 20 \text{ units} = 2 \text{ units}$$

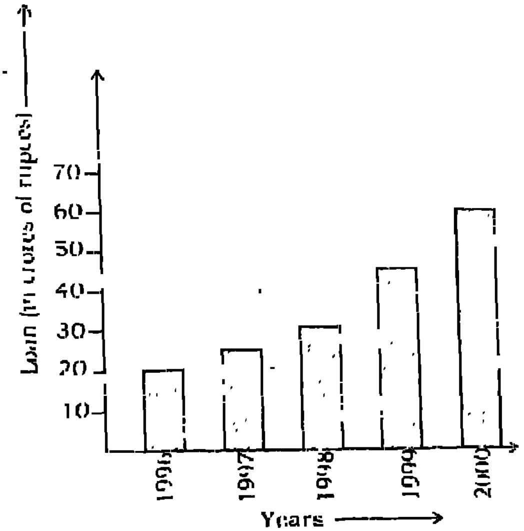
$$1997 : \frac{1}{10} \times 25 \text{ units} = 2.5 \text{ units}$$

$$1998 : \frac{1}{10} \times 30 \text{ units} = 3 \text{ units}$$

$$1999 : \frac{1}{10} \times 45 \text{ units} = 4.5 \text{ units}$$

$$2000 : \frac{1}{10} \times 60 \text{ units} = 6 \text{ units}$$

Step 6 : We draw five bars of equal width and of heights obtained in Step 5 above, the corresponding years marked on the horizontal axis, with equal spacing between them as shown in Fig.16.6.



Scale : One unit of graph paper along vertical axis = 10 crores of rupees

Fig.16.6 : Bar graph of the loans (in crores of rupees) given by a bank during the years 1996-2000.

Thus, Fig.16.6 is the required bar graph.

Example 4 : Draw a bar graph for information given in Example 1.

Solution : We go through the following steps :

Step 1 : Take a sheet of graph paper and draw two lines perpendicular to each other. Mark 'time intervals' along horizontal axis and 'number of vehicles' along vertical axis.

Step 2 : In this case, we choose 200 vehicles = 1 unit of the graph paper.

Step 3 : We calculate the heights of various bars as follows :

$$6 \text{ am} - 7 \text{ am} = \frac{1}{200} \times 50 \text{ units} = \frac{1}{4} \text{ unit}$$

$$7\text{am} - 8\text{am} = \frac{1}{200} \times 650 \text{ units} = 3\frac{1}{4} \text{ units}$$

$$8\text{am} - 9\text{am} = \frac{1}{200} \times 1650 \text{ units} = 8\frac{1}{4} \text{ units}$$

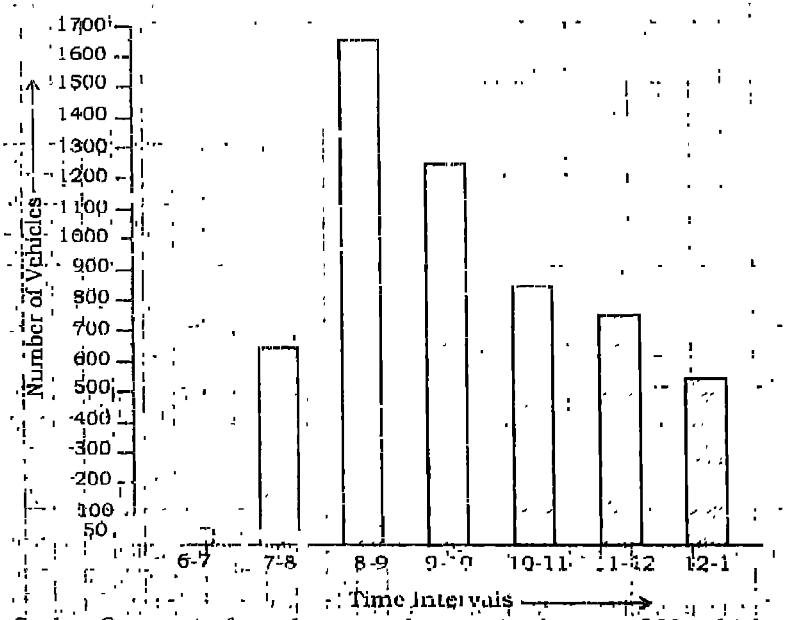
$$9\text{am} - 10\text{am} = \frac{1}{200} \times 1250 \text{ units} = 6\frac{1}{4} \text{ units}$$

$$10\text{am} - 11\text{am} = \frac{1}{200} \times 850 \text{ units} = 4\frac{1}{4} \text{ units}$$

$$11\text{am} - 12\text{noon} = \frac{1}{200} \times 750 \text{ units} = 3\frac{3}{4} \text{ units}$$

$$12\text{noon} - 1\text{pm} = \frac{1}{200} \times 550 \text{ units} = 2\frac{3}{4} \text{ units}$$

Step 4 : We draw the bars of equal width and of heights obtained in Step 3 corresponding to the various time intervals marked on the horizontal axis, with equal spacing between them as shown in Fig.16.7.



Scale . One unit of graph paper along vertical axis = 200 vehicles

Fig 16.7 : The bar graph shows the number of vehicles passing through a crossing during different time intervals

Thus, Fig.16.7 is the required bar graph.

EXERCISE 16.1

1. The bar graph given in Fig 16 8 represents the circulation of newspapers (dailies) in a town in six languages (the figures are approximated to hundreds). Study the bar graph and answer the following questions ·

   (i) Find the number of newspapers read in Hindi, Punjabi, Urdu, Marathi and Tamil.
   (ii) Find the excess of the number of newspapers read in Hindi over those of English.
   (iii) Name the language in which the least number of newspapers are read.
   (iv) Write in increasing order, the number of copies of newspapers read in different languages

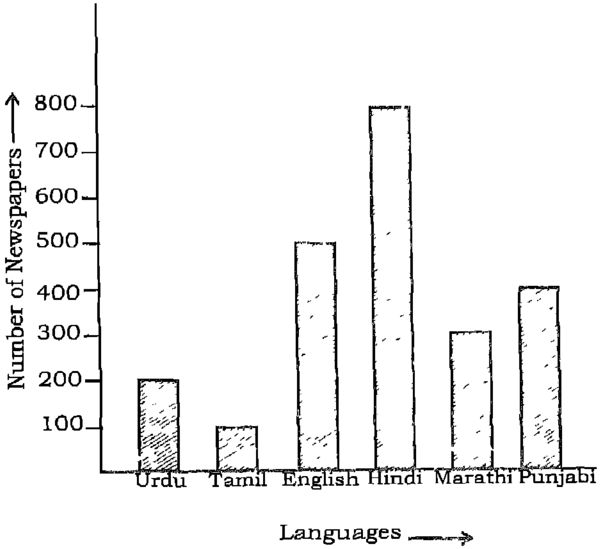


Fig.16.8: Circulation of newspapers in a town in six languages.

2. The number of bulbs sold by a shopkeeper on six consecutive days is as follows

Day	*Sun*	*Mon*	*Tues*	*Wed*	*Thurs*	*Fri*
Number of bulbs sold	55	32	30	25	10	20

Draw a bar graph to represent the above information.

3. Record the heights of ten students in your class and construct the bar graph to represent the information you have collected.

4. The number of persons in various age groups in a town is given in the following table :

Age group	1-14	15-29	30-44	45-59	60-74	75-89
Number of persons	1400	1200	1100	1000	950	300

Draw a bar graph to represent the above information and answer the following questions ;

(i) What is the percentage of the youngest age group persons over those in the oldest age group?

(ii) Find out the population of the town in all these age groups.

5. Marks obtained out of 100 in mathematics paper by 100 students are given in the following table :

Marks	9	19	58	61	75
Number of students	25	12	40	13	10

Represent the above information by a bar graph and answer the following questions ;

(i) Every student getting the highest marks is entitled to get a prize of Rs 10. How much money is required for distributing the prizes ?

(ii) Students obtaining the lowest marks have to solve 5 problems per day. How many problems in all will be solved by the students per day?

6. The heights (in cm) of 50 students of Class XI of a particular school are given as follows:

Height (in cm)	144	150	155	157	164
Number of students	7	8	17	13	5

Construct a bar graph to represent the above information Read the graph and answer the following questions .

(i) What per cent of the total number of students have their heights more than 150 cm?

(ii) How many students have heights above 150 cm but below 160 cm ?

Things to Remember

1. Pictorial representation of the numerical data using picture symbols is called a pictograph of the data.

2. A bar graph is a pictorial representation of the numerical data by a number of bars of uniform width, erected horizontally or vertically, with equal spacing between them.

3. The uniform width of the bars and the uniform gap between them is suitably chosen, keeping in view the given information (data) and the space available for the diagram.

4. It is not necessary to make the bar graph on a graph paper, but from the point of view of convenience and accuracy, one should do so
5. A bar graph should have a title. The chosen scale should also be mentioned below the graph

As History Tells Us

Initially, the word 'Statistics' used to mean simply the collection of information in the form of numerical data by the kings (or governments) on different aspects such as population, property, wealth, etc of their respective States. These data were useful to the kings in assessing the real situations in their respective States and consequently, in taking possible steps such as imposition of taxes and levies, utilization of the man power and natural resources for the welfare of the people of the State

The existence of this practice of collecting numerical data in ancient India is evident from the fact that during the reign of *Chandragupta Maurya* (324 BC - 300 BC), there was a very good system of collecting such data specially in regard to the births and deaths. During Akbar's reign (1556 - 1605 AD), *Raja Todarmal*, the then Land and Revenue Minister, maintained good records of land and agricultural statistics. In *Ain-i-Akbari* (आइन–ई–अकबरी) written by *Abul Fazl* (in 1596-97 AD), a detailed account of the administrative and statistical surveys conducted during that period can be found.

Thus, originally statistics was confined only to the affairs of the state However, with the passage of time, its scope was widened and it began to include collection of numercial data from almost every sphere of life (such as exports and imports, marriages and divorces, the daily maximum and minimum temperatures, cost of living index, etc.), and presentation of data in tabular and pictorial forms. By the end of the nineteenth century, the scope of statistics was further widened and it began to concern itself not only with the collection and presentation of data but also with interpretation and drawing of inferences from the data so obtained.

Data processing is an important activity to-day in the field of computers Powerful computers are engaged in processing trillion of trillions pieces of data day and night. Applications like weather prediction are made possible because of such data processing only, and this processing uses the basic principles of the subject known as Statistics in a very big way.

ANSWERS

EXERCISE 1.1

1. (i) 12 (ii) -27 (iii) 99 (iv) 1 (v) -67

2. (i) 23 (ii) 53 (iii) -1000 (iv) 101 (v) -167

3. (i) $\frac{8}{9}$ (ii) $\frac{-44}{46}$ (iii) $\frac{107}{51}$ (iv) $\frac{15}{2}$

4. (i) $\frac{1}{5}$ (ii) $\frac{-1}{5}$ (iii) $\frac{-1}{4}$ (iv) $\frac{1}{4}$ (v) $\frac{3}{8}$ (vi) $\frac{11}{107}$

5. (i) $\frac{5}{20}$ (ii) $\frac{9}{36}$ (iii) $\frac{-20}{-80}$ (iv) $\frac{-25}{-100}$ (v) $\frac{10000}{40000}$

6. (i) $\frac{-56}{-140}$ (ii) $\frac{154}{385}$ (iii) $\frac{-750}{-1875}$ (iv) $\frac{500}{1250}$ (v) $\frac{-6250}{-15625}$

7. (i) 90 (ii) 3 (iii) 12 (iv) 160

8. (i) $\frac{15}{18}$ and $\frac{14}{18}$ (ii) $\frac{8}{12}, \frac{10}{12}$ and $\frac{7}{12}$ (iii) $\frac{64}{80}, \frac{68}{80}, \frac{46}{80}$ and $\frac{55}{80}$

(iv) $\frac{120}{168}, \frac{63}{168}, \frac{108}{168}$ and $\frac{160}{168}$

9. (i) $\frac{2}{7}$ (ii) $\frac{2}{7}$ (iii) $\frac{2}{5}$ (iv) $\frac{-2}{7}$

10. (i) F (ii) T (iii) F (iv) T (v) F
(vi) T (vii) F (viii) T (ix) F (x) F

EXERCISE 1.2

2. Q represents $\frac{3}{4}$, R represents $\frac{3}{2}$ and S represents $\frac{9}{4}$; T represents $\frac{-5}{2}$.

3. (ii) **4.** (i) $\frac{3}{11}$ (ii) $\frac{-5}{8}$ (iii) $\frac{-7}{12}$ (iv) $\frac{-3}{-7}$

5. (i) $\frac{5}{-7}$ (ii) $\frac{6}{13}$ (iii) $\frac{16}{-5}$ (iv) $\frac{4}{-3}$

6. (i) $<$ (ii) $>$ (iii) $=$ (iv) $>$

7. (i) $\frac{4}{11}, \frac{3}{11}$ (ii) $\frac{5}{8}, \frac{3}{4}$ (iii) $\frac{7}{12}, \frac{5}{8}$ (iv) $\frac{4}{9}, \frac{3}{7}$

8. (i) $\frac{4}{7}, \frac{5}{7}$ (ii) $\frac{6}{13}, \frac{7}{13}$ (iii) $\frac{16}{5}$, 3 (iv) $\frac{4}{3}, \frac{8}{7}$

9. (i) $>$ (ii) $<$ (iii) $=$ (iv) $>$

10. (i) F (ii) F (iii) F (iv) T (v) F (vi) F (vii) T

EXERCISE 2.1

1. (i) $\frac{10}{7}$ (ii) $\frac{1}{13}$ (iii) $-\frac{5}{17}$ (iv) -1

2. (i) $-\frac{1}{36}$ (ii) $\frac{82}{99}$ (iii) $-\frac{26}{57}$ (iv) $-\frac{43}{78}$

5. (i) $\frac{37}{15}$ (ii) $-\frac{86}{63}$

EXERCISE 2.2

1. (i) $\frac{29}{75}$ (ii) $-\frac{17}{72}$ (iii) $\frac{29}{63}$ (iv) $\frac{1}{195}$

2. (i) $\frac{1}{4}, -\frac{1}{4}$, No (ii) $\frac{3}{8}, -\frac{3}{8}$, No (iii) $\frac{1}{66}, -\frac{1}{66}$, No

3. $-\frac{41}{7}$ **4.** $\frac{35}{38}$ **5.** $\frac{103}{72}$ **6.** $\frac{41}{33}$

7. (i) $-\frac{119}{36}, -\frac{143}{36}$, No (ii) $\frac{67}{63}, \frac{76}{63}$, No

8. (i) $\frac{19}{18}$ (ii) $\frac{41}{72}$ (iii) $-\frac{1}{10}$ (iv) $-\frac{35}{72}$

9. (i) $-\frac{5}{26}$ (ii) $-\frac{9}{14}$ (iii) $\frac{34}{9}$ (iv) $\frac{77}{23}$

10. (i) F (ii) T (iii) T (iv) T

EXERCISE 2.3

1. (i) $\frac{6}{55}$ (ii) $-\frac{6}{35}$ (iii) $-\frac{11}{81}$ (iv) $-\frac{3}{5}$ (v) $-\frac{3}{34}$

(vi) $-\frac{17}{21}$ (vii) $\frac{2}{3}$ (viii) $-\frac{24}{13}$ (ix) $\frac{2}{3}$ (x) $\frac{1}{10}$

2. (i) $-\frac{16}{5}$ (ii) $-\frac{1}{12}$ (iii) $\frac{42}{1}$ (iv) $-\frac{3}{7}$

4. (ii) $\frac{5}{9}\times(-4)$ (iii) $\frac{-5}{8}\times\frac{3}{11}$ (iv) $\frac{-3}{-7}\times(-6)$

6. (ii) $\left(\frac{1}{2}\times\frac{3}{4}\right)\times\frac{-5}{13}$ (iii) $(-4\times(-6))\times\left(-\frac{7}{11}\right)$ (iv) $\left(-\frac{2}{9}\times\frac{4}{5}\right)\times\frac{3}{7}$

9. (ii) $\frac{-3}{8}\times\frac{-6}{11}+\frac{-3}{8}\times\frac{4}{9}$ (iii) $6\times\frac{5}{13} + 6\times\frac{-3}{4}$

(iv) $\frac{2}{3}\times\frac{-5}{7} - \frac{2}{3}\times\frac{4}{5}$ (v) $\frac{-4}{7}\times\frac{3}{4}-\frac{-4}{7}\times\frac{-2}{3}$

10. (i) T (ii) F (iii) F (iv) F (v) F (vi) F (vii) T (viii) F

EXERCISE 2.4

1. (i) $\frac{1}{17}$ (ii) $\frac{-1}{19}$ (iii) $\frac{13}{8}$ (iv) $\frac{-29}{13}$

6. (i) $\frac{20}{3}$ (ii) $-\frac{1}{6}$ (iii) $\frac{7}{8}$ (iv) $-\frac{15}{4}$

7. $\frac{10}{3}$ **8.** $\frac{4}{3}$ **10.** (i) F (ii) F (iii) F (iv) T (v) T

EXERCISE 2.5

1. (i) $-\frac{13}{2}$ (ii) $-\frac{85}{84}$ (iii) 0 (iv) 0

2. (i) $\frac{4}{5}, \frac{9}{10}, 1$ (ii) $-\frac{100}{77}, \frac{-90}{77}, \frac{-80}{77}$

(iii) $\frac{6}{7}, \frac{8}{7}, \frac{10}{7}$ (iv) $-\frac{577}{546}, \frac{-136}{273}, \frac{11}{182}$

3. No. $\frac{3}{5}, \frac{27}{70}$ *or* $\frac{3}{5}, \frac{57}{70}$ **4.** No. $\frac{4}{9}, \frac{133}{198}$ *or* $\frac{4}{9}, \frac{43}{198}$

5. (i) $\frac{-1}{3}, \frac{-1}{5}, 0, \frac{1}{3}, \frac{1}{5}$ (ii) $\frac{-1}{2}, \frac{-1}{4}, 0, \frac{1}{4}, \frac{1}{2}$

6. (i) $\frac{4789}{2604}$ (ii) $\frac{-23389}{12056}$ **7.** (i) $-\frac{71}{12}$ (ii) $\frac{71}{12}$

8. $(x \times y)^{-1} = x^{-1} \times y^{-1}$ **9.** $(x \div y)^{-1} = x^{-1} \div y^{-1}$

10. (i) F (ii) F (iii) T (iv) T (v) F
(vi) T (vii) T (viii) F (ix) F (x) F

EXERCISE 3.1

1. (i) 3.75 (ii) 2.04 (iii) 2. 405 (iv) 8.025 (v) -0.5 (vi) -0.65 (vii) 0.085 (viii) -1.3125

2. (ii), (v) and (viii). The denominator is a multiple of 2 and / or 5 only.

3. (i) $.\overline{90}$ (ii) $.\overline{153846}$ (iii) $.1\overline{8}$ (iv) $.\overline{1}$ (v) $.\overline{027}$ (vi) $-6.\overline{6}$ (vii) $3\ \overline{142857}$ (viii) $-1.\overline{3}$

4. (i) F (ii) T (iii) F (iv) T (v) F

EXERCISE 3.2

1. (i) $\frac{14}{5}$ (ii) $\frac{37}{1000}$ (iii) $\frac{-3}{4}$ (iv) $\frac{-69}{8}$

(v) $\frac{-79}{100}$ (vi) $\frac{7543}{1000}$ (vii) $\frac{48}{5}$ (viii) $\frac{-47}{8}$

2. (i) $\frac{25}{2}$ (ii) $\frac{22}{5}$ (iii) $\frac{389}{50}$ (iv) $\frac{1147}{50}$ (v) $\frac{4387}{200}$ (vi) $\frac{184071}{10000}$

3. (i) $\frac{12821}{1000}$ (ii) $\frac{1421}{100}$ (iii) $\frac{1574}{125}$ (iv) $\frac{-81}{50}$ (v) $\frac{400127}{500}$

4. (i) $\frac{11424}{25}$ (ii) $\frac{924}{125}$ (iii) $\frac{3969}{400}$ (iv) $\frac{1008}{25}$ (v) $\frac{27}{1000000}$

5. (i) $\frac{18}{5}$ (ii) $\frac{3027}{50}$ (iii) $\frac{729}{625000}$ (iv) 70 (v) 1526

EXERCISE 4.1

1. (i) $\frac{9}{49}$ (ii) $\frac{243}{1024}$ (iii) $\frac{16}{81}$ (iv) $\frac{-125}{729}$

2. (i) $\frac{3}{625}$ (ii) $\frac{-1}{12}$ (iii) 6561 (iv) 864

3. (i) $\left(\frac{1}{3}\right)^5$ (ii) $-\left(\frac{4}{27}\right)^2$ (iii) $-\left(\frac{5}{11}\right)^4$ (iv) $\left(\frac{7}{4}\right)^4$

4. (i) $\frac{3}{16}$ (ii) $\frac{-9}{16}$ (iii) $\frac{15}{8}$ (iv) 125

5. (i) $-\frac{1}{243}$ (ii) $\frac{256}{81}$ (iii) 15625 (iv) $\frac{9}{49}$

6. (i) $\frac{1}{27}$ (ii) $\frac{32}{16807}$ (iii) $\frac{625}{81}$ (iv) $\frac{121}{169}$

7. (i) $\left(\frac{3}{4}\right)^2$, $\frac{10}{16}$, $\frac{11}{16}$, $\frac{12}{16}$, … , $\frac{35}{16}$, there can be many more

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EXERCISE 4.2

1. (i) 7 (ii) 11 (iii) 11 (iv) 3
(v) 6 (vi) 4 (vii) 6 (viii) 4

2. (i) $\frac{32}{243}$ (ii) $\frac{9}{16}$ (iii) $\frac{1}{16}$ (iv) $\frac{1}{64}$

3. (i) $\left(\frac{5}{2}\right)^8$ (ii) $\left(\frac{11}{3}\right)^{24}$ (iii) $\left(\frac{3}{4}\right)^3$
(iv) $\left(\frac{5}{4}\right)^5$ (v) $\left(\frac{2}{3}\right)^8$ (vi) $\left(\frac{3}{4}\right)^{12}$

4. (i) F (ii) T (iii) T (iv) F (v) T (vi) F

EXERCISE 4.3

1. (i) $\frac{1}{8}$ (ii) $\frac{1}{81}$ (iii) 256 (iv) $\frac{243}{32}$ (v) $\frac{25}{9}$ (vi) $\frac{-343}{64}$ (vii) $\frac{256}{81}$

2. (i) $\left(\frac{3}{2}\right)^5$ (ii) $\left(\frac{1}{2}\right)^8$ (iii) $\left(\frac{1}{4}\right)^2$ (iv) $\left(\frac{2}{3}\right)^6$ (v) $\left(-\frac{1}{14}\right)^3$ (vi) $\left(\frac{1}{2}\right)^{10}$

3. (i) 2^{-1} (ii) $\left(\frac{1}{2}\right)^{-12}$ (iii) $\left(\frac{1}{4}\right)^{-7}$ (iv) $\left(\frac{-5}{6}\right)^{-6}$ (v) $\left(\frac{1}{5}\right)^{-4}$ (vi) $\left(\frac{1}{3}\right)^{7}$

4. (i) 1 (ii) 1 (iii) 1 (iv) 1 (v) 3

(vi) 1 (vii) 0 (viii) 0

5. 3^8 **6.** $(-4)^7$ **7.** -2 **8.** -1 **9.** $\frac{27}{32}$ **10.** $\frac{729}{64}$

11. $\frac{32}{729}$ **13.** (i) T (ii) T (iii) T (iv) T (v) T

EXERCISE 4.4

1. (i) 9.8×10^{8} (ii) 9.7×10^{-11} (iii) 55×10^{-14} (iv) 10.7×10^{9}
(v) 60.2×10^{22}, 6.02×10^{23}

2. (i) 0.0000065 (ii) 0.0000000089 (iii) 56146929
(iv) 5800000000000 (v) 1001000000

3. (i) 1.05×10^{6} (ii) 1.353×10^{9}, 1.361×10^{9}
(iii) 1.027×10^{9}, 5.312×10^{8}, 4.958×10^{8} (iv) 1×10^{-6}

EXERCISE 5.1

1. (i)

x	–	–	–	21	–
y	–	27	39	–	75

(ii)

x	–	–	–	–	11	–
y	2.5	–	7.5	10	–	12

(iii)

x	–	–	–	8	–
y	0.4	2	–	–	128

2.

Time	–	–	7	–	155
Height of the balloon (in metres)	36	–	–	300	–

3. 30 stamps **4.** 480 tools **5.** Rs 7.30 **6.** 64 bottles

7. 1.925cm **8.** Rs 157500 **9.** 108 words

EXERCISE 5.2

1. (i) Inverse variation (ii) Inverse variation (iii) Direct variation

2. (i) Possible (ii) Possible (iii) Not possible (iv) Possible

3. (i), (ii), (iii) are possible; (iv) not possible

4. 10 km **5.** $3\frac{1}{4}$ hours **6.** 16 litres **7.** 4 km/h **8.** $5\frac{1}{3}$ hours

9. 3000 persons **10.** 10 weeks **11.** 16 km/h **12.** 50 cycles

13. 160 persons

14.

Pressure (in cm)	–	–	90.00	–	60.00
Volume (in ml)	–	11.25	–	8	–

15.

Volume (V)	–	–	–	25	12.5
Absolute temperature (T)	–	360	450	–	–

16. 12 days **17.** 2 days **18.** 12 hours **19.** 24 seconds

20. 250m **21.** 72km/h **22.** 50.4 seconds **23.** 120 km/h

EXERCISE 6.1

1. (i) Rs 300 (ii) Rs 10000 (iii) Rs 2000

2. Rs 4000 **3.** 240 days **4.** 176400 **5.** 800 **6.** Rs 7000

7. 2000, 2160 **8.** 1150, 460 **9.** 20 **10.** 1200

EXERCISE 6.2

1. (i) C.P = Rs 450 , S.P. = Rs 540, Profit = 20%

(ii) S.P. = Rs 3038, Loss = 2%, Overhead expenses = Rs 100

(iii) C.P. = Rs 30000, Profit = Rs 6000, Profit = 20%

(iv) Profit = Rs 72 , S.P. = Rs 972 , Purchased at Rs 400

(v) C.P. = Rs 250, Loss = Rs 15 , S.P. = Rs 235

2. 20 **3.** Loss 10% **4.** Profit = 25% **5.** 25% **6.** Profit = 35%

7. C.P. = Rs 900 **8.** C.P. = Rs 12800 **9.** Profit = 20%

10. Rs 1108.80 **11.** Rs 187.50 **12.** Gain 10%

EXERCISE 6.3

1. (i) Rs 180 (ii) Rs 700 (iii) Rs 60, Rs 260

(iv) Rs 210, Rs 810 (v) Rs 28 (vi) Rs 66.50

(vii) Rs 6.75, Rs 186.75 (viii) 12.5% (ix) Rs 8.00, Rs 488.00

(x) $T = 2\frac{1}{2}$ years, A = Rs 792 (xi) R = 10%, I = Rs 150

2. (i) S.I. = Rs 192, A = Rs 992 (ii) S.I. = Rs 162, A = Rs 612

(iii) S.I. = Rs 20, A = Rs 620

3. (i) Rs 400 (ii) Rs 350 (iii) Rs 9600

4. (i) $2\frac{1}{2}$ years (ii) $\frac{1}{2}$ year (iii) 7 years

5. (i) 7.5% (ii) 13% (iii) 18%

6. Rs 1050 **7.** Rs 12060 **8.** A = Rs 603 20, S.I. = Rs 83.20

9. Principal = Rs 2200 **10.** P = Rs 7100 **11.** P = Rs 1800

12. Rs 1600 **13.** $R = 12\frac{1}{2}\%$ **14.** R = 22% **15.** T = 8 years

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EXERCISE 7.1

1. (i) $6a^2$ (ii) $8a^5$ (iii) $-42\,ab^2c$

(iv) $-49\,x^3yz$ (v) $\frac{4}{17}x^3y^5$ (vi) $\frac{2}{3}\,a^3b^4c^5$

(vii) $0.72\,p^3q^4$ (viii) $9\,9\,pqr$

2. (i) $60\,a^3b^3c^3$ (ii) $300\,p^3q^3$ (iii) $105\,b^3c^3d^2$

(iv) $\frac{2}{5}x^6y^6z^6$ (v) $7\,986\,p^2q^2r^2$ (vi) $0.54\,a^4b^3c^6$

3. (i) a^{100} (ii) $a^{150}b^{100}c^{100}d^{101}$ (iii) $-\frac{1}{9}a^3b^3c^3$ (iv) 0

4. $-50a^6$ **5.** $-30\,x^3y^4$ **6.** $-1600\,a^{10}b^5$; -50

7. $-81.92\,x^5y^5$, $-2\,56$

9. (i) $66\,a^3x^5$ (ii) $-\frac{1}{5}p^3q^3r^3$

10. (i) x and yz ; y and xz ; z and xy ; 1 and xyz

(ii) a and ab , b and a^2 ; 1 and a^2b

(iii) 7 and pq ; p and $7q$; q and $7p$; 1 and $7pq$

11. (i) It represents area of a rectangle with sides $3x$ and $4x$

(ii) It represents area of a rectangle with sides 7 and $5p$

12. (i) 280 (ii) 60 **13.** (i) $a^2b^2c^2d^2$ (ii) $x^4y^3z^6$

14. (i) $15a^2 + 18a$ (ii) $-6a^3 - 26a^2$

(iii) $8x^3 - 12x^2y^2$ (iv) $-15p^3qr^2 - 35p^2q^2r^2$

15. (i) $\frac{3}{8}x^2y^2 + \frac{1}{2}x^3y$ (ii) $\frac{7}{3}a^3b^5 - 2a^4b^4$

16. (i) $3.3a^3b - 1.65a^2b^2$; 19.8

(ii) $-0.81a^2b^2 + 1.08a^4$; 14.04

17. (i) $-3x^3y + 3xy^3$ (ii) $\frac{1}{2}x^5y^3z^3 + \frac{1}{2}x^3y^5z^3$

18. (i) $a^2 - b^2$ (ii) $2a^2 + ab - b^2$

(iii) $a^3 + b^3$ (iv) $-3p^2 - 51p$

EXERCISE 7.2

1. (i) $12x^2 + 64x + 45$ (ii) $3x^2 - 17x - 56$

(iii) $\frac{3}{4}a^5 + 7a^3b + \frac{1}{6}a^2b^2 + \frac{14b^3}{9}$ (iv) $6.25a^2 - 5.29b^2$

(v) $6p^2q^2 + 13pq^3 + 6q^4$

2. (i) $8x^2 - 6x - 35$ (ii) $7x^2 + 6xy - y^2$

(iii) $a^2b^2 + a^3 + b^3 + ab$ (iv) $p^3 - p^2q - pq^2 + q^3$

3. (i) $8x^3y + 22x^2y^2 + 15xy^3$ (ii) $a^5b^3 + 3a^5 + 5b^3 + 19$

(iii) $a^4 + ab^3c^3d^3 + a^3bcd + b^4c^4d^4$ (iv) $-4m^2n^2 + 6mn + 8n^3$

(v) $t^4 - s^6$ (vi) $4ac$

4. (i) $x^4 - 25y^4$ (ii) $-2x^4 - 4x^2y^2 - 2y^4$

(iii) $x^3 + 5x^2 - 5x - 20$ (iv) $5x^2 + 3x - 2xy + 21y - 14y^2$

(v) $3x^2 + 4xy - y^2$

5. (i) $2x^2 - 5xy + 7x + 14y - 18y^2$ (ii) $\frac{3}{2}x^2 - \frac{163}{8}xy + 16x - 4y + 5y^2$
(iii) $x^3 + x^3y + x^2y + xy^2 + xy^3 + y^3$ (iv) $a^4 + a^3b + a^3c - ab^3 - b^3c - b^4$

6. (i) $2.25x^2 + 4.5x - 16y^2 - 12y$ (ii) $m^2p^2 - m^2n^2 - n^4 + p^4$

7. (i) $x^3 + y^3$ (ii) $10x^2 + 3xy^2 - y^2 - y^3$ (iii) $x^2 + 2xy^2 + y^3$

EXERCISE 7.3

1. (i) $x^2 + 6x + 9$ (ii) $4y^2 + 20y + 25$
(iii) $\frac{4}{25}p^2 + \frac{12}{5}p + 9$ (iv) $1.21m^2 + 4.62m + 4.41$
(v) $9a^2 + 24ab + 16b^2$ (vi) $\frac{1}{4}x^2 + \frac{3}{4}xy + \frac{9}{16}y^2$

2. (i) $a^2 - 10a + 25$ (ii) $4a^2 - 2a + \frac{1}{4}$ (iii) $\frac{25}{4}x^2 - 35x + 49$
(iv) $49a^2 - 126ab + 81b^2$ (v) $x^4 - 2x^2y^2 + y^4$ (vi) $a^2b^2 - 2ab^2c + b^2c^2$

3. (i) $36x^2 - 49$ (ii) $\frac{1}{4}x^2 - 1$ (iii) $9a^2 - 49b^2$
(iv) $144y^2 - 121x^2$ (v) $b^4 - a^4$ (vi) $4x^6 - 81y^6$

4. (i) $a^2 - 10a + 25$ (ii) $4a^2 + 28a + 49$ (iii) $9a^4 + 24a^2b + 16b^2$
(iv) $36x^4 - 60x^2y + 25y^2$ (v) $49y^4 - 112x^3y^2 + 64x^6$
(vi) $16m^6 + 88m^3n^3 + 121n^6$

5. (i) $36x^2 - 64y^2$ (ii) $b^6 - 9a^4$ (iii) $\frac{4}{9}m^4 - \frac{9}{64}n^4$ (iv) $2.89p^6 - 1.44a^6$

6. (i) $a^4 - 2a^2b^2 + b^4$ (ii) $a^6 + 2a^3b^3 + b^6$
(iii) $4x^2 + 12xy^3 + 9y^6$ (iv) $49p^6 - 70p^3a^2 + 25a^4$

7. (i) $8x^2 + 50$ (ii) $18p^2 + 128q^2$
(iii) $6600\,mn$ (iv) $-\frac{1}{50}r^2t^2$

8. (i) $a^2b^2 + b^2c^2$ (ii) $m^4 + m^2n^4$

9. (i) $(3x - 7)^2$ (ii) $(89p + 5q)^2$

10. (i) 5041 (ii) 8464 (iii) 10609
(iv) 3481 (v) 9801 (vi) 982081

11. (i) 9975 (ii) 6396 (iii) 89991

12. (i) 200 (ii) 2540 (iii) 2760

13. (i) 62 (ii) 143 (iii) 12

EXERCISE 8.1

1. (i) $2x$ (ii) $7pq$ (iii) $3abc$ (iv) b^2
(v) $5a$ (vi) abc (vii) 2

2. (i) $2a^2$ (ii) $3x^2y^2$ (iii) $-4a^2$ (iv) 4

3. (i) $7(x + 3)$ (ii) $6(p - 2)$ (iii) $a(a + 2)$ (iv) $5x(2 + x)$
(v) $a(7a + 2)$ (vi) $3xy(x + 2y)$
(vii) $4m(5m^2 - 4)$ (viii) $10pq\,(2p + a)$

4. (i) $2x^2(-3x^2 + x - 5)$ (ii) $10ab\,(4a^2b - a^2 + 2b^2)$
(iii) $5(2a^3 - 3b^3 + 4c^3)$ (iv) $a(a^2bc + 4b^3 + 41a^2)$

5. (i) $(5a + b)\,(5a - b)$ (ii) $(7p + 6)\,(7p - 6)$
(iii) $(2a^2b^2 + 3pq)\,(2a^2b^2 - 3pq)$ (iv) $(ab + 3)\,(ab - 3)$
(v) $(2n + 5m)\,(2n - 3m)$

6. (i) $(a + 4)\,(a + 4)$ (ii) $(b - 5)\,(b - 5)$
(iii) $4(a - 1)\,(a - 1)$ (iv) $(5x + 3)\,(5x + 3)$
(v) $(7a + 6b)\,(7a + 6b)$ (vi) $(11m - 4n)\,(11m - 4n)$

7. (i) $(x+y)(x+9)$ (ii) $(2x+1)(5y+2)$

(iii) $(x-4)(5x+2y)$ (iv) $(2y-3)(3x-2)$

8. (i) $x(px+q)$ (ii) $16x^5(x^2-3)$

(iii) $7(x^2+3y^2)$ (iv) $2(5x+6y)(5x-6y)$

(v) $7(3x+4y)(3x-4y)$ (vi) $-4pq$

(vii) $2x(x^2+y^2+z^2)$ (viii) $3a^2(1-3b-9ac)$

9. (i) $(x-z+y)(x-z-y)$ (ii) $(5a+c+7b)(5a+c-7b)$

(iii) $(p^2+q^2)(a+b)$ (iv) $(a+1)(b+1)$

10. (i) $(a^2+b^2)(a+b)(a-b)$ (ii) $(m^2+16)(m+4)(m-4)$

(iii) $(x^2+y^2+z^2+2yz)(x+y+z)(x-y-z)$

(iv) $z(2x-z)(2x^2+z^2-2xz)$

EXERCISE 9.1

1. $x=\frac{-14}{3}$

2. $x=-1$

3. $x=\frac{4}{5}$

4. $x=\frac{31}{2}$

5. $x=-6$

6. $m=\frac{7}{5}$

7. $p=\frac{17}{9}$

8. $t=\frac{5}{4}$

9. $x=-2$

10. $y=\frac{2}{3}$

11. $x=-96$

12. $w=\frac{-5}{18}$

13. $x=\frac{183}{10}$ *or* 18.3

14. $y=\frac{-33}{2}$ *or* -16.5

15. $z=42$

16. $x=18$

17. $x=\frac{9}{2}$

18. $x=\frac{2}{3}$

19. $y=\frac{6}{5}$

20. $p=\frac{1}{8}$

EXERCISE 9.2

1. 21 **2.** 30 **3.** 8

4. 46, 49 **5.** 7, 8 and 9 **6.** 4 years and 24 years

7. Length = 23 m, breadth = 19 m **8.** 5 years

9. No. of 100 rupee prizes = 19, No. of 25 rupee prizes = 44

10. No. of 10 rupee note = 5, No. of 50 rupee note = 4

11. Length = 10 cm, breadth = 4 cm **12.** 40, 15 and 25

13. 25 boys **14.** No. of 500 rupee prizes = 75, No. of 100 rupee prizes = 125

15. Rs 480000 **16.** $\frac{7}{4}$ **17.** Length = 100 m, breadth = 50 m

18. $14\frac{1}{2}kg$ **19.** 4 hours **20.** 36

EXERCISE 10.1

13. (i) cannot be constructed (iv) cannot be constructed
(v) cannot be constructed

EXERCISE 10.2

1. $\angle A = \angle B$ **2.** EF = DE **3.** $\angle Q = 45°, \angle P = 90°$

4. $\angle C = 50°$ **5.** $\angle Q = 108°$ **6.** XY = YZ

7. BC = 4cm

8. (i) $x = 30°$ (ii) $y = 60°$ (iii) $z = 60°$

9. (i) $\angle PRQ = 80°$ (ii) $\angle PQR = 50°$

10. (i) $\angle ABC = 70°, \angle ACB = 40°$ (ii) $x = 110°, y = 140°$

12. (i) $\angle ABC = 60° = \angle ACB$
(ii) $\angle DBC = 70° = \angle DCB$
(iii) $\angle ABD = 130°, \angle ACD = 130°$; Yes

13. $\angle P = 90°, \angle Q = 45° = \angle R$

14. (i) No (ii) $\angle R > \angle Q$

(iii) The greater angle is opposite the longer side.

15. (i) No (ii) $AC > AB$

(iii) The longer side is opposite the greater angle.

EXERCISE 10.3

1. $AB = 15$ cm

2. (i) $c^2 = 6.25 \text{ cm}^2$ (ii) $c^2 = 42.25 \text{ cm}^2$ (iii) $c^2 = 380.25 \text{ cm}^2$

(iv) $c^2 = 2500 \text{ cm}^2$ (v) $c^2 = 676 \text{ cm}^2$

3. 7 cm 4. 8 m 5. 10 cm

6. (ii), (iii), (iv), (vi) 7. ΔABC is right angled at B

8. No, $AB^2 > AC^2 + BC^2$ is true

EXERCISE 10.4

1. (i) line segment, perpendicular (ii) orthocentre

(iii) Outside (iv) AC and BC

(v) AC (vi) Concurrent

(vii) Centroid (viii) AB

(ix) Interior

2. (ii) Yes 3. Point C 4. (i) Yes (ii) outside

5. Yes, DR = RE, FR is a median of the ΔDEF

EXERCISE 10.5

1. (i) Concurrent (ii) Perpendicular bisectors

(iii) Concurrent (iv) Angle bisectors (v) $\angle A$

3. No 4. Yes 5. Yes

EXERCISE 11.1

1. $\angle D = \angle Y$

2. (i), (iii), (vi), (vii), (viii)

3. (i) $\angle R = \angle X$ (ii) $QR = ZX$ (iii) $\angle P = \angle Y$
(iv) $QP = ZY$ (v) $\angle Q = \angle Z$ (vi) $RP = XY$

4. (i) $\Delta PQR \cong \Delta XZY$ (iii) $\Delta ABC \cong \Delta RQP$ (v) $\Delta AOB \cong \Delta DOC$
(vi) $\Delta PQR \cong \Delta RSP$ (vii) $\Delta BAD \cong \Delta CAD$

5. $AD = AD$; yes

6. (i) Yes, Alternate interior angles
(ii) Yes
(iii) $AB = DC$, $\angle BAC = \angle DCA$ and $AC = AC$

7. (iii)

EXERCISE 11.2

1. $AB = EF$

2. (i) $\Delta ABC \cong \Delta FDE$ (ii) $\Delta ABC \cong \Delta ANM$
(iv) $\Delta ABC \cong \Delta XYZ$ (vi) $\Delta XYZ \cong \Delta QPZ$

3. $AD = AD$

4. (i) Yes, vertically opposite angles
(ii) Yes
(iii) $AO = BO$, $\angle A = \angle B$ and $\angle AOC = \angle BOD$
(iv) Yes, since $\Delta AOC \cong \Delta BOD$

5. (i) Yes
(ii) $\angle BAD = \angle CAD$, $AD = AD$ and $\angle ADB = \angle ADC$
(iii) Yes, because $\Delta ADB \cong \Delta ADC$

EXERCISE 11.3

1. $AC = DE$

2. (i) $\Delta ABC \cong \Delta FDE$ (ii) $\Delta PQR \cong \Delta XZY$ (iii) $\Delta ABC \cong \Delta EFD$

(iv) $\Delta ABO \cong \Delta QPO$ (v) $\Delta DEF \cong \Delta NML$

(vi) $\Delta ABD \cong \Delta CDB$

3. $AC = AC$ **4.** $AD = AD$

5. $\Delta PQR \cong \Delta TRQ$

EXERCISE 11.4

1. $AC = YZ$

2. (i) $\Delta ABC \cong \Delta FDE$ (iii) $\Delta RPQ \cong \Delta YXZ$

(v) $\Delta RPQ \cong \Delta QDR$ (vi) $\Delta CAO \cong \Delta DBO$

3. (i) $AC = AC$ (ii) Yes, since $\Delta ABC \cong \Delta ADC$

4. (i) Yes

(ii) $AB = AC$, $\angle ADB = \angle ADC = 90^\circ$ and $AD = AD$

(iii) Yes, since $\Delta ABD \cong \Delta ACD$

5. $BD = CE$, $\angle BDC = \angle CEB = 90^\circ$ and $BC = CB$

MISCELLANEOUS EXERCISE

1. (a) (i) FD (ii) DE (iii) FE

(iv) $\angle F$ (v) $\angle D$ (vi) $\angle E$

(b) $\angle R$ (c) EF (d) SAS

(e) ASA

2. (i) SSS, $\Delta ABC \cong \Delta DFE$

(ii) RHS, $\Delta XYZ \cong \Delta DFE$

(iii) ASA, $\Delta ABC \cong \Delta CDA$

(iv) SAS, $\Delta PTQ \cong \Delta STR$

3. (i) ASA (ii) SSS (iii) RHS

(iv) ASA (v) SAS (vi) ASA

(vii) SSS (viii) ASA (ix) SAS

4. SAS ; 40°

EXERCISE 12.1

1. (i) 4 (ii) 4 (iii) 4 (iv) 2

2. ΔACD, ΔACB; Two triangles

3. Four triangles; ΔAOB, ΔBOC, ΔCOD, ΔDOA

4. (i) Four — PQ and QR ; QR and RS ; RS and SP ; SP and PQ.

(ii) Two — PQ and RS ; PS and QR.

(iii) Four — $\angle P$ and $\angle Q$; $\angle Q$ and $\angle R$; $\angle R$ and $\angle S$; $\angle S$ and $\angle P$.

(iv) Two — $\angle P$ and $\angle R$; $\angle Q$ and $\angle S$.

5. (i) Vertices — A, B, C, D.

(ii) Angles — $\angle A$, $\angle B$, $\angle C$, $\angle D$.

(iii) Diagonals — AC and BD.

(iv) Adjacent sides — AB and BC ; BC and CD ;
CD and DA ; DA and AB.

(v) Adjacent angles — $\angle A$ and $\angle B$; $\angle B$ and $\angle C$;
$\angle C$ and $\angle D$; $\angle D$ and $\angle A$.

(vi) Opposite sides — AB and CD ; BC and DA.

(vii) Opposite angles — $\angle A$ and $\angle C$; $\angle B$ and $\angle D$.

6. Yes, one point.

EXERCISE 12.2

1. 115°
2. 70°
3. 95°
4. 90°
5. No ; Sum of the four angles is greater than 360°.
6. 45°, 75°, 105°, 135°
7. 36°, 72°, 108°, 144°

EXERCISE 13.1

4. 60° ; Angles in the same segment
5. All angles are equal (90°)
6. $\angle ACD = \angle APD = 75^\circ$
7. $\angle ACB = 90^\circ$; $\angle ABC = 30^\circ$
8. $\angle RQS = 50^\circ$; $\angle RPS = 50^\circ$
9. $\angle DFE = 45^\circ$; $\angle FED = 85^\circ$
10. 70°

EXERCISE 14.1

1. 1750 m^2
2. 1900 m^2
3. 1176 m^2
4. 30 cm^2
5. (i) 4.84 m^2 (ii) 1.60 m^2
6. (i) 30 m^2 (ii) Rs 750
7. (i) 116 m^2 (ii) Rs 278.40
8. (i) 301 cm^2 (ii) Rs 745.20
9. 4975 m^2, Rs 522375
10. Rs 15594
11. (i) 71.9 m^2 (ii) 338.1 m^2
12. (i) 441 m^2 (ii) Rs 48510
13. (i) 308 m^2 (ii) 2700 m^2
14. (i) 588 m^2 (ii) 6612 m^2

EXERCISE 15.1

1. (i) Cuboid : Butter box ; Lunch box ; Chalk box ; Air conditioner.
   (ii) Cube : Ice cubes ; Sugar cubes ; Dice ; Cubical blocks

2. Faces : PQRS ; TUVW ;
PQUT ; SRVW ;
PSWT ; QRVU ;
Edges : PS ; PQ ; QR ; RS ;
TU ; UV ; VW ; WT ;
PT ; SW ; UQ ; RV ;

3. AB = CD = GH = x ; AD = BC = EH = y ; AE = BF = DH = z

4. ABFE ; BCGF ; CDHG ; DAEH **5.** AE or BF or CG or DH

6. (i) BFGC (ii) ABCD ; EFGH ; ABFE ; CDHG
(iii) ABFE ; ABCD (iv) HG ; HE ; HD

7. ABCD ; ABFE ; AEHD. Vertex G ; Yes ; there are four diagonals
AG ; BH, CE, DF

8. Cuboid **9.** Cube

10. (i) 6 (ii) 12 (iii) Edge
(iv) Dimensions (v) 4 (vi) Cube
(vii) 8 (viii) Vertex (ix) Congruent squares
(x) 3 (xi) 90° (xii) 4

EXERCISE 15.2

1. (i) For painting or white-washing a room.
(ii) For making a box or a trunk.
(iii) For painting an almirah.

2. 654 cm^2 **3.** 544 cm^2 **4.** 4750 cm^2

5. (i) 726 cm^2 (ii) 8.64 m^2 (iii) 4374 cm^2

6. 1350 cm^2

7. (i) 120640 cm^2 (ii) Rs 241.28 **8.** Rs 110800

9. (i) $9.16\ m^2$ (ii) Rs 458 **10.** $400\ m^2$

11. $66\ m^2$ **12.** $278\ m^2$ **13.** Rs 20880

14. Rs 30000 **15.** 100 Bricks

EXERCISE 15.3

1. (i) $960\ cm^3$ (ii) $1760.22\ cm^3$ (iii) $1672704\ mm^3$

(iv) $445.295\ m^3$ (v) $5\ 04\ m^3$ (vi) $56250\ cm^3$

2. (i) $3375\ mm^3$ (ii) $1953\ 125\ cm^3$ (iii) $17.576\ m^3$

(iv) $5.088448\ m^3$

3. 3 cm **4.** $180\ cm^3$ **5.** 135000 *l*

6. 6 m **7.** 4 m **8.** Rs 4320

9. 2 m **10.** 3 days **11.** $432\ cm^3$

12. 16000 crates **13.** 4000 blocks **14.** 27000 blocks

15. (i) 8 times (ii) $\left(\frac{1}{8}\right)^{th}$ times (iii) 27 times

EXERCISE 16.1

1. (i) Hindi = **800** ; **Punjabi** = 400 ; Urdu = 200 , Marathi = 300 , Tamil = 100

(ii) 300 (iii) Tamil

(iv) Tamil = 100, Urdu = 200, Marathi = 300 , Punjabi = 400,
English = 500, Hindi = 800

4. (i) $466\frac{2}{3}\%$ (ii) 5950

5. (i) Rs 100 (ii) 125

6. (i) 70% (ii) 30

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